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NOTE

GRAPH EXPONENTIATION AND NEIGHBORHOOD RECONSTRUCTION

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Abstract

Any graph G admits a neighborhood multiset $\mathscr{N}(G) = \{N_G(x) \mid x \in V(G)\}$ whose elements are precisely the open neighborhoods of G. We say G is neighborhood reconstructible if it can be reconstructed from $\mathscr{N}(G)$, that is, if $G \cong H$ whenever $\mathscr{N}(G) = \mathscr{N}(H)$ for some other graph H. This note characterizes neighborhood reconstructible graphs as those graphs G that obey the exponential cancellation $G^{K_2} \cong H^{K_2} \Longrightarrow G \cong H$.

Keywords: neighborhood reconstructible graphs, graph exponentiation.

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Our graphs are finite and may have loops, but not parallel edges. The open neighborhood of a vertex x of a graph G is $N_G(x) := \{y \in V(G) \mid xy \in E(G)\}$. Notice that $x \in N_G(x)$ if and only if $xx \in E(G)$, that is, there is a loop at x.

To any graph G there is an associated *neighborhood multiset* $\mathscr{N}(G) = \{N_G(x) \mid x \in V(G)\}$ whose elements are the open neighborhoods of G. It is possible that $\mathscr{N}(G) = \mathscr{N}(H)$ but $G \ncong H$. Figure 1 shows the simplest instance of this. Here $G \ncong H$ but $\mathscr{N}(G) = \big\{\{0\}, \{1\}\big\} = \mathscr{N}(H)$. Figure 2 shows a more complex and interesting example.

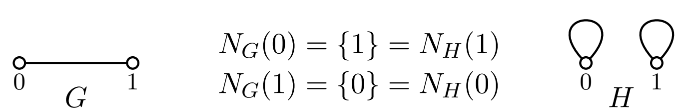

Figure 1. Two non-isomorphic graphs with the same neighborhood multiset.

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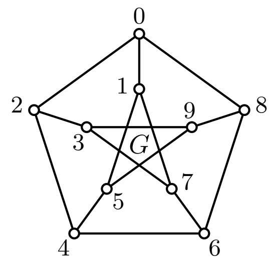

$N_G(0)$	$=$	$\{1,2,8\}$	$=$	$N_H(0)$
$N_G(1)$	$=$	$\{0,5,7\}$	$=$	$N_H(1)$
$N_G(2)$	$=$	$\{0,3,4\}$	$=$	$N_H(8)$
$N_G(3)$	$=$	$\{2,7,9\}$	$=$	$N_H(6)$
$N_G(4)$	$=$	$\{2,5,6\}$	$=$	$N_H(9)$
$N_G(5)$	$=$	$\{1,4,9\}$	$=$	$N_H(5)$
$N_G(6)$	$=$	$\{4,7,8\}$	$=$	$N_H(3)$
$N_G(7)$	$=$	$\{1,3,6\}$	$=$	$N_H(7)$
$N_G(8)$	$=$	$\{0,6,9\}$	$=$	$N_H(2)$
$N_G(9)$	$=$	$\{3,5,8\}$	$=$	$N_H(4)$

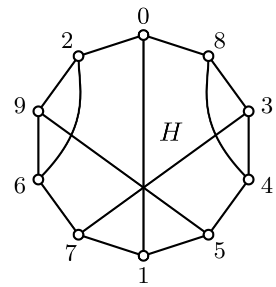

Figure 2. The Petersen graph is not neighborhood reconstructible. It is paired here with a different graph that has the same neighborhood multiset. Example from Mizzi [5, § 3.9].

A graph G is called *neighborhood reconstructible* if $\mathscr{N}(G) = \mathscr{N}(H)$ implies $G \cong H$ for any graph H with $V(H) = V(G)$. Figure 2 shows that the Petersen graph is not neighborhood reconstructible. Aigner and Triesch [1] attribute the neighborhood reconstruction problem to Sós [9]. They note that deciding if a graph is neighborhood reconstructible is NP-complete.

Given graphs G and K, the *graph exponential* G^K is the graph whose vertex set is the set of all functions $V(K) \to V(G)$, where two functions f, g are adjacent precisely if $f(x)g(y) \in E(G)$ for all $xy \in E(K)$. (See [6, 8].) If $V(K) = \{v_1, \ldots, v_n\}$, then a function $f : V(K) \to V(G)$ can be identified with an n-tuple $f = (x_1, \ldots, x_n) \in V(G)^n$ signifying $f(v_i) = x_i$.

We are interested exclusively in G^{K_2}. Note $V\big(G^{K_2}\big) = V(G) \times V(G)$, and two functions (x_1, x_2) and (y_1, y_2) are adjacent if and only if $x_1y_2 \in E(G)$ and $x_2y_1 \in E(G)$. That is,

$$E\big(G^{K_2}\big) = \big\{(x_1, x_2)(y_1, y_2) \mid x_1y_2 \in E(G) \text{ and } x_2y_1 \in E(G)\big\}.$$

See Figure 3, which shows that $G^K \cong H^K$ does not necessarily imply $G \cong H$.

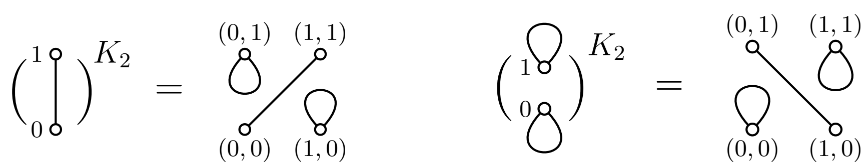

Figure 3. Two exponentials G^{K_2} and H^{K_2}. This shows $G^K \cong H^K$ may not imply $G \cong H$.

Actually, the conditions under which $G^K \cong H^K$ implies $G \cong K$ are not fully understood today. (The issue is further complicated by the fact that there are at least two definitions of graph exponentiation; compare [4].) This note links one instance of this *exponential cancellation* to neighborhood reconstruction. Our main result is that G is neighborhood reconstructible if and only if $G^{K_2} \cong H^{K_2}$ implies $G \cong H$ for all graphs H. To understand why we might expect this, consider Proposition 1 below, whose proof is almost automatic. (Figures 1 and 3 illustrate Proposition 1.)

Proposition 1. *If G and H are two graphs on the same vertex set and $\mathscr{N}(G) = \mathscr{N}(H)$, then $G^{K_2} \cong H^{K_2}$.*

Proof. Say $\mathscr{N}(G) = \mathscr{N}(H)$. As G and H have the same neighborhood multiset, there is a bijection $\varphi : V(G) \to V(H)$ for which $N_G(x) = N_H(\varphi(x))$ for each $x \in V(G)$. (Such map φ is unique if no two vertices of G have the neighborhood; otherwise there is more than one φ.) The bijection $\lambda : V\left(G^{K_2}\right) \to V\left(H^{K_2}\right)$ where $\lambda(x, y) = (\varphi(x), y)$ is an isomorphism. Indeed,

$$\begin{aligned}(x,y)(u,v) \in E\left(G^{K_2}\right) &\iff v \in N_G(x) \text{ and } y \in N_G(u)\\ &\iff v \in N_H(\varphi(x)) \text{ and } y \in N_H(\varphi(u))\\ &\iff \big(\varphi(x), y\big)\,\big(\varphi(u), v\big) \in E\big(H^{K_2}\big)\\ &\iff \lambda(x,y)\,\lambda(u,v) \in E\big(H^{K_2}\big).\end{aligned}$$

■

We will use this proposition in the proof of our main result. We will also need the *direct product* of graphs: $G \times H$ is the graph whose vertex set is the set Cartesian product $V(G \times H) = V(G) \times V(H)$, and whose edges are

$$E(G \times H) = \left\{(x,y)(x'y') \mid xx' \in E(G) \text{ and } yy' \in E(H)\right\}.$$

See Chapter 8 of [2] for a survey of the direct product.

For a positive integer k, the *direct power* G^k is $G \times \cdots \times G$ (k factors). Any square G^2 admits a *mirror automorphism* $\mu : G^2 \to G^2$ of order 2, where $\mu(x, y) = (y, x)$. From the definitions it is immediate that

$$(1) \qquad (x,y)(u,v) \in E\big(G^2\big) \quad \text{if and only if} \quad (x,y)\mu(u,v) \in E\big(G^{K_2}\big),$$

$$(2) \qquad (x,y)(u,v) \in E\big(G^{K_2}\big) \quad \text{if and only if} \quad \mu(x,y)(u,v) \in E\big(G^2\big).$$

Recall the following two results (by Lovász) concerning direct powers and products. (They are Theorems 2 and 5, respectively, in [7].)

Proposition 2. *If $G^k \cong H^k$ for a positive integer k, then $G \cong H$.*

Proposition 3. *If $G \times K \cong H \times K$, then there is an isomorphism $G \times K \to H \times K$ of form $(x, y) \mapsto (\lambda(x, y), y)$ for some map $\lambda : G \times K \to H$.*

Actually, we will only need a weaker instance of Proposition 3, one that is easy to prove from scratch. If $G \times K_2 \cong H \times K_2$, then there exists an isomorphism $G \times K_2 \to H \times K_2$ of form $(x, y) \mapsto (\lambda(x, y), y)$.

We are ready for our main theorem.

Theorem 4. *A graph G is neighborhood reconstructible if and only if the exponential cancellation law $G^{K_2} \cong H^{K_2} \Rightarrow G \cong H$ holds for any graph H.*

Proof. Say the exponential cancellation law $G^{K_2} \cong H^{K_2} \Rightarrow G \cong H$ holds. Let $\mathscr{N}(G) = \mathscr{N}(H)$ for a graph H with $V(H) = V(G)$. Proposition 1 yields $G^{K_2} \cong H^{K_2}$, whence $G \cong H$. Thus G is neighborhood reconstructible.

Conversely, suppose G is neighborhood reconstructible. Say $G^{K_2} \cong H^{K_2}$ for some graph H. We must show $G \cong H$.

Put $V(K_2) = \{0, 1\}$. Take an isomorphism $\varphi : G^{K_2} \to H^{K_2}$. Using (1) and (2), observe that

$$\begin{aligned}(x,y)(u,v) \in E\big(G^2\big) &\iff (x,y)\,\mu(u,v) \in E\big(G^{K_2}\big)\\ &\iff \varphi(x,y)\,\varphi\mu(u,v) \in E\big(H^{K_2}\big)\\ &\iff \mu\varphi(x,y)\,\varphi\mu(u,v) \in E\big(H^2\big).\end{aligned}$$

From this we get an isomorphism $\Theta : G^2 \times K_2 \to H^2 \times K_2$ defined as

$$\Theta\big((x,y),\varepsilon\big) = \begin{cases} (\varphi\mu(x,y),\varepsilon) & \text{if } \varepsilon = 0,\\ (\mu\varphi(x,y),\varepsilon) & \text{if } \varepsilon = 1.\end{cases}$$

From $G^2 \times K_2 \cong H^2 \times K_2$ we get $G^2 \times K_2 \times K_2 \cong H^2 \times K_2 \times K_3$, yielding $(G \times K_2)^2 \cong (H \times K_2)^2$. By Proposition 2 we have $G \times K_2 \cong H \times K_2$. Then Proposition 3 guarantees an isomorphism $\theta : G \times K_2 \to H \times K_2$ having form

$$\theta(x,\varepsilon) = \begin{cases} (\lambda_0(x),\varepsilon) & \text{if } \varepsilon = 0,\\ (\lambda_1(x),\varepsilon) & \text{if } \varepsilon = 1\end{cases}$$

for two bijections $\lambda_0, \lambda_1 : V(G) \to V(H)$, which (by definition of the direct product) necessarily satisfy $xy \in E(G)$ if and only if $\lambda_0(x)\lambda_1(y) \in E(H)$.

Now form a graph H' on $V(G)$ whose edges are precisely $\lambda_1^{-1}(u)\lambda_1^{-1}(v)$ for each $uv \in E(H)$. Thus $\lambda_1^{-1} : H \to H'$ is an isomorphism.

We claim that $N_G(x) = N_{H'}\big(\lambda_1^{-1}\lambda_0(x)\big)$ for each $x \in V(G) = V(H')$. Note $y \in N_G(x)$ if and only if $xy \in E(G)$, if and only if $\lambda_0(x)\lambda_1(y) \in E(H)$, if and only if $\lambda_1^{-1}\lambda_0(x)\,\lambda_1^{-1}\lambda_1(y) \in E(H')$, if and only if $\lambda_1^{-1}\lambda_0(x)\,y \in E(H')$, if and only if $y \in N_{H'}\big(\lambda_1^{-1}\lambda_0(x)\big)$. Thus indeed $N_G(x) = N_{H'}\big(\lambda_1^{-1}\lambda_0(x)\big)$.

Consequently $\mathscr{N}(G) = \mathscr{N}(H')$, so $G \cong H'$ because G is neighborhood reconstructible. But $H' \cong H$, so $G \cong H$. ∎

The present note is a sequel to [3], which characterizes neighborhood reconstructible graphs as those graphs G which obey the cancellation law $G \times K \cong H \times K \Rightarrow G \cong K$ for all graphs H and K.

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