Note

Robust cycle bases do not exist for \(K_{n,n}\) if \(n \geq 8\)

Richard H. Hammack\(^a,^*\), Paul C. Kainen\(^b\)

\(^a\) Virginia Commonwealth University, Richmond, VA, USA
\(^b\) Georgetown University, Washington DC, USA

A B S T R A C T

A basis for the cycle space of a graph is said to be robust if any cycle \(Z\) of \(G\) is a sum \(Z = C_1 + C_2 + \cdots + C_k\) of basis elements such that (i) \((C_1 + C_2 + \cdots + C_{\ell-1}) \cap C_{\ell}\) is a nontrivial path for each \(2 \leq \ell < k\). Hence, (ii) each partial sum \(C_1 + C_2 + \cdots + C_{\ell}\) is a cycle for \(1 \leq \ell \leq k\). While complete graphs and 2-connected plane graphs have robust cycle bases, it is shown that regular complete bipartite graphs \(K_{n,n}\) do not have any robust cycle basis if \(n \geq 8\).

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1. Introduction

The problem of whether or not it is possible to find a graph with no robust basis has been open for nearly 20 years. We show that regular complete bipartite graphs \(K_{n,n}\) have no robust bases when \(n \geq 8\).

In the last five decades, cycle bases of graphs have been considered from novel perspectives. For instance, the minimum cycle basis problem asks for a cycle basis of smallest total length. Gleiss’s dissertation [4] attributes this problem to Stepanec [17], and Zykov [20] in the Russian literature. M. Plotkin [16], a chemist, defined a graph cycle as relevant if it is not a sum of shorter cycles. Vismara [19] showed that a cycle is relevant if and only if it belongs to some minimum cycle basis.

Different questions were raised by Dixon and Goodman [2]. Their article seems to be the first appearance (in print) of the concept of a weakly robust basis, which is a cycle basis satisfying only the second condition (ii) given in the abstract. They conjectured that the bases associated with spanning trees are weakly robust. However, Sysło [18] gave a counter-example. Twenty years later, Dogrusöz and Krishnamoorthy [3] argued that for a 2-connected plane graph, the Mac Lane basis (the set of boundary cycles of the bounded regions) is weakly robust. Also, Ostermeier et al. [15] showed that the set of \(C_4\)-subgraphs containing a given edge of \(K_{m,n}\) is weakly robust, and they gave a short proof of weak robustness for the Mac Lane basis.

The notion of a robust basis was formulated in [10], and applied to commutativity of diagrams. Also [10] proves that a robust basis of the complete graph \(K_n\) can be formed by taking all \(K_3\)-subgraphs containing a given vertex, and it notes that Mac Lane’s basis of a 2-connected plane graph is robust. An explicit proof is given in [12], which further shows that no repeated terms are needed in the robust sums.

A substantial literature on cycle bases has developed (see, e.g., [6–9,14]). Applications have included the analysis of random protein networks [13], energy models for RNA folding [5] and commutativity of algebraic diagrams [11].

The remainder of the paper is organized as follows. In Section 2, we review the relevant background; results are proved in Section 3. The last section is a discussion.

* Corresponding author.

E-mail addresses: rh hammack@vcu.edu (R.H. Hammack), kainen@georgetown.edu (P.C. Kainen).

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2. Definitions

The cycle space $\mathcal{C}(G)$ of a graph $G$ is the subset of the power set of $E(G)$ consisting of the subsets whose edge-induced subgraphs of $G$ have no vertices of odd degree, endowed with the structure of a vector space over the two-element field $F_2 = \{0, 1\}$. Addition is symmetric difference, and $\emptyset$ is the zero vector. Informally, one views $\mathcal{C}(G)$ as the set of spanning even-degree subgraphs of $G$, where the edgeless subgraph is zero. If $G$ has $c$ components, then $\mathcal{C}(G)$ has dimension $|E(G)| - |V(G)| + c$; see, e.g., Diestel [1, pp. 23–28]. A cycle in $G$ is a 2-regular connected subgraph of $G$. Because any even-degree subgraph of $G$ is the sum (possibly a trivial sum) of edge-disjoint cycles, $\mathcal{C}(G)$ is spanned by the cycles in $G$, and so has a basis whose elements are cycles. Such a basis for $\mathcal{C}(G)$ is called a cycle basis.

A cycle basis $\mathcal{B}$ of $\mathcal{C}(G)$ is a weakly robust basis if for each cycle $Z$ in $G$ there is a sequence $C_1, C_2, \ldots, C_k$ of elements of $\mathcal{B}$ (possibly with repetition) for which

$$Z = C_1 + C_2 + \cdots + C_k$$

and each partial sum $C_1 + C_2 + \cdots + C_i$ is a cycle for $1 \leq i \leq k$. The basis is called a robust basis if $(C_1 + C_2 + \cdots + C_i) \cap C_{i+1}$ is a nontrivial path for $1 \leq i \leq k - 1$. Here each summand is attached to the previous sum in a 1-cell, like a hinge. In such a case $C_1 + C_2 + \cdots + C_k$ is called a robust sum. In a robust basis, cycles are built by a sequence of attachments. Note that a robust basis is weakly robust. Acyclic graphs have empty bases, which are vacuously robust.

To see the difference between robust and weakly robust cycle bases, let $G$ be a Möbius ladder. Embed $G$ on a Möbius strip, and view the strip as a Möbius cap of the projective plane. Let $\mathcal{B}$ be the set of squares on the ladder, union a cycle in $G$ with non-trivial homotopy in the projective plane. Check that $\mathcal{B}$ is a weakly robust basis but not a robust basis. (The boundary cycle of the ladder, which is homotopically trivial on the projective plane, is not a robust sum of basis elements.)

It is not known whether some graph has no weakly robust basis. Indeed, for bipartite complete graphs, such a weakly robust basis does exist. Ostermeier et al. [15] showed that the basis defined below satisfies weak robustness.

As in [10], we construct a cycle basis for $K_{n,m}$ as follows. Fix an edge $ab$. For any edge $xy$ vertex-disjoint from $ab$, let $S_{xy}$ denote the $K_{2,2}$ = $C_4$-subgraph "square" induced by the set $\{a, b, x, y\}$ in $K_{n,m}$. The set of squares which contain $ab$,

$$\mathcal{X} := \mathcal{X}_{ab} := \{S_{xy} \mid xy \in E(K_{n,m} - \{a, b\})\},$$

is independent because each square $S_{xy}$ in $\mathcal{X}$ has the edge $xy$ that belongs to no other, and so is a basis, as $|\mathcal{X}| = (m - 1)(n - 1) = mn - (m + n) + 1$ is the dimension of the cycle space of $K_{n,m}$. The basis $\mathcal{X}$ is called the Kainen basis in [9,14,15]. It is shown in [15] that $\mathcal{X}$ is robust if $m \leq 4$ and $n \leq 5$, and that $\mathcal{X}$ is not robust if $m, n \geq 5$. We will shortly prove the following slightly stronger result. (See also the Discussion section below.)

**Proposition 1.** The basis $\mathcal{X}$ is robust if and only if $\min\{m, n\} \leq 4$.

This begs the question of whether or not any robust basis exists for $K_{m,n}$ when $\min\{m, n\} > 4$. We show the answer is "No" for $K_{n,n}$, with $n \geq 8$.

3. Results

This section depends on the following definition and proposition.

**Definition 1.** A cycle $C$ in a graph $G$ is contiguous with a Hamiltonian cycle $H$ in $G$ if at most one edge of $C$ is not an edge of $H$. Thus $C$ being contiguous with $H$ means that either $H = C$, or $H = D + C$, where $D$ is a cycle intersecting $C$ precisely at an edge $ab$. (See in Fig. 1.)

In [12], two cycles are called compatible if they intersect in a nontrivial path. The nonidentity case of contiguity constrains the compatibility in two ways: $C + D$ must be spanning while $C \cap D$ is a path of one edge.

**Proposition 2.** If $\mathcal{B}$ is a robust cycle basis for a graph $G$, and $H$ is a Hamiltonian cycle in $G$, then there is some $C \in \mathcal{B}$ that is contiguous with $H$. 

Proof. Let \( \mathcal{R} \) be a robust cycle basis for \( G \) and let \( H \) be a Hamilton cycle in \( G \). Then either \( H \in \mathcal{R} \) (and we are done), or \( H = C_1 + C_2 + C_3 + \cdots + C_k \), with each summand in \( \mathcal{R} \), and where \( (C_1 + C_2 + C_3 + \cdots + C_{k-1}) \cap C_k \) is a non-trivial path for each \( 1 < \ell \leq k \). Let \( D = C_1 + C_2 + C_3 + \cdots + C_k \). Then \( H = D + C_k \) and \( D \cap C_k \) is a non-trivial path. But this path cannot have any internal vertices, for then they would not appear on the Hamiltonian cycle \( H \). Thus \( D \cap C_k \) is an edge, so \( C_k \) is contiguous with \( H \). □

In what follows, we regard the vertices in one partite set of \( K_{n,m} \) as colored black, and those in the other as colored white. We now prove Proposition 1, that the Kainen basis \( \mathcal{X} \) of \( K_{n,m} \) is robust if and only if \( \min(m, n) \leq 4 \).

Proof of Proposition 1. The statement is vacuously true if \( \min(m, n) = 1 \), so assume \( 2 \leq \min(m, n) \). For arbitrary \( m, n \), the longest cycle in \( K_{m,n} \) has length \( 2 \cdot \min(m, n) \). Thus it suffices to show that any cycle \( Z \) of length at most \( 8 \) in any \( K_{m,n} \) is a robust sum of elements from \( \mathcal{X} \).

First suppose that \( Z = x_0x_1x_2 \cdots x_{2k-1} \) passes through neither \( a \) nor \( b \), and without loss of generality, say that \( a \) and \( x_0 \) are in opposite partite sets. Note that \( Z = S_{x_0x_1} + S_{x_1x_2} + S_{x_2x_3} + \cdots + S_{x_{2k-2}x_{2k-1}} \) is a robust sum because \( S_{x_0x_1} + S_{x_1x_2} + \cdots + S_{x_{2k-2}x_{2k-1}} \cap S_{x_{2k-2}x_{2k-1}} = P \), where \( P \) is the path \( abx_0 \) if \( \ell \) is odd and \( \ell < 2k - 1 \), while \( P = ax_\ell \) if \( \ell \) is even. And finally, if \( \ell = 2k - 1 \), then \( P \) is the path \( P = x_0abx_{2k-1} \).

Next, suppose \( Z \) passes through both \( a \) and \( b \). If \( ab \) happens to be an edge of \( Z \), then take a path \( xayb \) in \( Z \) and note that \( Z + S_{xy} \) is a robust sum equaling a cycle \( Z' \) that misses both \( a \) and \( b \). Then \( Z = Z' + S_{xy} \), and we can proceed by decomposing \( Z' \) as in the previous paragraph.

On the other hand, if \( Z \) passes through both \( a \) and \( b \), but \( ab \) is not an edge of \( Z \), then because the even cycle \( Z \) has length no greater than \( 8 \), it contains a path \( xayb \). Then \( Z + S_{xy} = Z' \) is a robust sum where \( Z' \) contains the edge \( ab \). Then \( Z = Z' + S_{xy} \), and we decompose \( Z' \) as in the previous paragraph.

Finally, suppose \( Z \) contains only one of \( a \) or \( b \) (say \( a \)). Take a path \( xay \) on \( Z \). Notice \( Z + S_{xy} = Z' \) is a robust sum and \( Z' \) is a cycle containing the edge \( ab \). Then \( Z = Z' + S_{xy} \), and we decompose \( Z' \) as before.

To see that \( \mathcal{X} \) is not robust when \( \min(m, n) > 4 \), let \( Z \) be a cycle of length 10 in \( K_{n,m} \), for which \( a \) and \( b \) are at distance 5 from each other in \( Z \). (As shown on the left in Fig. 2.) Suppose to the contrary that \( \mathcal{X} \) is robust.

If \( m = n = 5 \), then \( Z \) is Hamiltonian. Notice that in this case no element of \( \mathcal{X} \) is contiguous with \( Z \), contradicting Proposition 2. In general, for \( n \geq m \geq 5 \), the cycle \( Z \) is a robust sum

\[
Z = S_{x_1y_1} + \cdots + S_{x_ty_t} + S_{x,y}
\]

whose last term is some basis element \( S_{x,y} \). The right of Fig. 2 shows the penultimate partial sum \( S_{x_1y_1} + \cdots + S_{x_ty_t} \). Observe that no matter the edge \( xy \), the partial sum \( S_{x_1y_1} + \cdots + S_{x_ty_t} \) is not a cycle, so the sum (1) cannot be robust, contrary to assumption. □

Having seen that the Kainen basis is robust only for \( \min(m, n) \leq 4 \), we now prove that in fact there does not exist any robust basis for \( K_{n,m} \) when \( n \geq 8 \). Our approach uses a counting argument, involving Hamiltonian cycles, based on a well-known lemma. (The corresponding result for directed Hamiltonian cycles appears in Sequence A010790 in the Online Encyclopedia of Integer Sequences, http://oeis.org/.) For completeness, we give a short proof.

Lemma 1. The graph \( K_{n,n} \) has \( \frac{n}{2} \) \((n-1)!\) Hamiltonian cycles.

Proof. Fix a black vertex \( a \) of \( K_{n,n} \). We will build a Hamiltonian cycle \( H \) through \( a \) by first choosing two white vertices \( x \) and \( y \) to be \( H \)-neighbors of \( a \). There are \( \binom{n}{2} \) ways to make this choice. Continuing the cycle from \( a \) through \( y \), there are \( n - 1 \) choices for the black vertex after \( y \), then \( n - 2 \) for the next white, one, then \( n - 2 \) for a black, then \( n - 3 \) for a white, then \( n - 3 \) for a black, etc. (See Fig. 3.) Thus the number of Hamiltonian cycles in \( K_{n,n} \) is \( \binom{n}{2} (n-1)(n-2)(n-3)^2 \cdots 1^2 = \frac{n}{2} (n-1)! \). □
Theorem 1. If $K_{n,n}$ has a robust cycle basis, then $n \leq 7$.

Proof. Say $K_{n,n}$ has a robust cycle basis $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$, where $p = n^2 - 2n + 1 = (n - 1)^2$, which is the dimension of the cycle space of $K_{n,n}$. As Proposition 1 asserts that such a robust basis exists when $n = 2, 3, 4$ we assume henceforward that $n \geq 4$.

In what follows we first show that $n \leq 8$. Further analysis will then improve this to $n < 8$. We proceed via a sequence of claims.

Claim 1. If $C_i \in \mathcal{C}$ has length $2k < 2n$, then it is contiguous with $2k ((n - k)!)^2$ Hamiltonian cycles. And (obviously) if $C_i$ has length $2k = 2n$ then it is contiguous with exactly one Hamiltonian cycle, namely itself.

To prove this, take such a $C_i$ of length $2k < 2n$. Select an edge $ab$ of $C_i$, with $a$ black and $b$ white. Let us count the ways to extend $C_i - ab$ to a Hamiltonian cycle $H$. (That is, so that $C_i$ is contiguous with $H$ and $ab$ is the only edge of $C_i$ not on $H$.) We first run an edge from $a$ to any of the $n - k$ white vertices in $V(H) - V(C_i)$. From that vertex, we may extend an edge to any of the $(n - k)$ black vertices in $V(H) - V(C_i)$. Then we extend to any of the $n - k - 1$ remaining white vertices, then to any of the remaining $n - k - 1$ black vertices, etc. (See Fig. 4.) In this way we see that $C_i$ is contiguous with $((n - k)!)^2$ Hamiltonian cycles $H$ in such a way that $ab$ is the only edge of $C_i$ not in $H$. As $ab$ is one of $2k$ edges in $C_i$, it follows that $C_i$ is contiguous with $2k((n - k)!)^2$ Hamiltonian cycles.

Claim 2. If $3 \leq k \leq n$ and $4 \leq n$, then $2k ((n - k)!)^2 \leq 2 ((n - 2)!)^2$.

For $k = 3$ the inequality holds by elementary algebraic inspection (using $n \geq 4$). Now assume $k > 3$. Notice that $2k \leq 2 ((k - 2)!)^2$ because beyond $k = 3$ the linear left-hand side is overtaken by the right-hand side. Using this with the fact $k \leq n$, we get

$$2k \leq 2 ((k - 2)!)^2 = 2(k - 2)^2(k - 3)^2(k - 4)^2 \cdots (k - (k - 1))^2 \leq 2(n - 2)^2(n - 3)^2(n - 4)^2 \cdots (n - (k - 1))^2 = 2((n - 2)!)^2.$$

Comparing the first and last expressions yields $2k((n - k)!)^2 \leq 2((n - 2)!)^2$, confirming the claim.

Next we establish an upper bound on the number of Hamiltonian cycles in $K_{n,n}$. By Claim 1, with $k = 2$, any square in $\mathcal{C}$ is contiguous with $4((n - 2)!)^2$ Hamiltonian cycles. Also, by Claim 1, if $C_i \in \mathcal{C}$ is not a square (that is, if it has length $2k$ with $k > 2$), then $C_i$ is contiguous with $2k((n - k)!)^2$ Hamiltonian cycles, and, by Claim 2, this does not exceed $2((n - 2)!)^2$.

Conversely, Proposition 2 shows that each Hamiltonian cycle is counted since there is an element in $\mathcal{C}$ contiguous with it.

Let $x$ be the number of elements of $\mathcal{C}$ that are squares; let $y$ be the number of elements that are not squares (that is, have length greater than 4). By the above remarks, the total number of Hamiltonian cycles in $K_{n,n}$ does not exceed

$$x \cdot 4 ((n - 2)!)^2 + y \cdot 2 ((n - 2)!)^2.$$

Using Lemma 1,

$$\frac{n}{2} ((n - 1)!)^2 \leq x \cdot 4 ((n - 2)!)^2 + y \cdot 2 ((n - 2)!)^2. \tag{2}$$

Thus $n(n - 1)^2 \leq 8x + 4y$. As $(n - 1)^2 = |\mathcal{C}| = x + y$, we get $n(x + y) \leq 8x + 4y$. Then

$$n \leq 4 + \frac{4x}{x+y} = 4 + \frac{4x}{|\mathcal{C}|}. \tag{3}$$

From this it follows that $n \leq 8$. However, one more step improves the result to $n < 8$.

Claim 3. Suppose that for any Hamiltonian cycle $H$ of $K_{n,n}$, there is at most one square in $\mathcal{C}$ that is contiguous with $H$. Then $n = 4$. Otherwise $n \leq 7$. 

Fig. 4. Constructing Hamiltonian cycles that $C_i$ is contiguous with.
Fig. 5. The five ways that two squares on $K_{n,n}$ can intersect.

Fig. 6. The squares in $B$.

To prove this, we count the ways two squares $S, S'$ in $K_{n,n}$ can meet. Fig. 5 shows the five possibilities for the intersection: empty, a single vertex, a path of length 1, a path of length 2, or (in the last case) two nonadjacent vertices. The figure shows that in the first four cases $S$ and $S'$ are contiguous with a common Hamiltonian cycle.

But inspection reveals that in the last case (intersection at two vertices), $S$ and $S'$ are not contiguous with a common Hamiltonian cycle. By assumption, any two squares of $B$ intersect in this way. Thus the squares in $B$ are arranged as in Fig. 6, that is, any two of them intersect at a fixed set $\{c, d\}$ of vertices in the same partite set.

As each square uses two vertices of one of the partite sets, the number $x$ of squares in $B$ is no more than $\frac{n^2}{2}$. Note that for $n \geq 4$, we have $x \leq \frac{n^2}{2} < \frac{1}{2}(n - 1)^2 = \frac{1}{4}|B|$. Substituting this in inequality (3) yields $n \leq 4$, so $n = 4$.

Finally, if two squares in $B$ are contiguous with the same Hamiltonian cycle, then we have double counted Hamiltonian cycles in the inequality (2), so it becomes strict, and the inequality (3) yields $n < 4 + 4 = 8$. □

4. Discussion

Proposition 1 says that $K_{n,n}$ has a robust basis for $n \leq 4$. By Proposition 1, no such basis exists for $n \geq 8$. The question is open for $n = 5, 6,$ and 7.

The robust span of some family $\mathcal{F}$ of cycles in a graph $G$ is the family $\rho(\mathcal{F})$ of all cycles with a robust sum from $\mathcal{F}$. Take $G = K_{n,n}$, with $n \geq 8$. For any basis $\mathcal{B}$ one has $\rho(\mathcal{B}) \subseteq \text{Cyc}(G)$, where $\text{Cyc}(G)$ denotes the set of all cycle-subgraphs of $G$. However, the basis $\mathcal{X}$ will now be shown to be iteratively robust in that $\rho^k(\mathcal{X}) = \text{Cyc}(G)$ for sufficiently large $k$, where the superscript on $\rho$ means iterating the operation. Hence,

$$\rho(\mathcal{X}) \subseteq \rho^2(\mathcal{X}).$$

(4)

To prove the iterative robustness of $\mathcal{X}$, recall that a cycle $Z$ in $G$ is geodesic if each pair of points in $Z$ is joined by a $G$-geodesic path completely contained within $Z$. It is shown in [12, Thm. 6.1] that for any graph $G$, $\rho^k(\mathcal{F}) = \text{Cyc}(G)$ for large enough $k$, where $\mathcal{F}$ denotes the family of all geodesic cycles in $G$. For $K_{n,n}$ a cycle is geodesic if and only if it has length 4. But the proof of Proposition 1 shows that each cycle of length at most 8 is a robust sum of cycles from $\mathcal{X}$ and hence $\mathcal{X}$ is iteratively robust.
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