A New View of Hypercube Genus

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Abstract. Beineke, Harary, and Ringel discovered a formula for the minimum genus of a torus in which the n-dimensional hypercube graph can be embedded. We give a new proof of the formula by building this surface as a union of certain faces in the hypercube's 2-skeleton. For odd dimension n, the entire 2-skeleton decomposes into $(n-1)/2$ copies of the surface, and the intersection of any two copies is the hypercube graph.

1. INTRODUCTION. What graphs can be drawn on what surfaces without crossed edges? Kuratowski's theorem [**3**, Theorem 6.18] implies that the complete bipartite graph $K_{m,n}$ (Figure 1) cannot be drawn on the sphere (or plane) without crossed edges if $\min\{m, n\} \geq 3$. One can try to draw $K_{3,3}$ on the sphere, but will always fail. However, it can be drawn on the torus, as shown in Figure 1.

The **genus** of a graph G, denoted $\gamma(G)$, is the least integer g for which G can be drawn on a closed, connected, orientable surface of genus g without edges crossing. (A sphere has genus 0; a surface with g holes has genus g.) Thus $\gamma(K_{3,3}) = 1$.

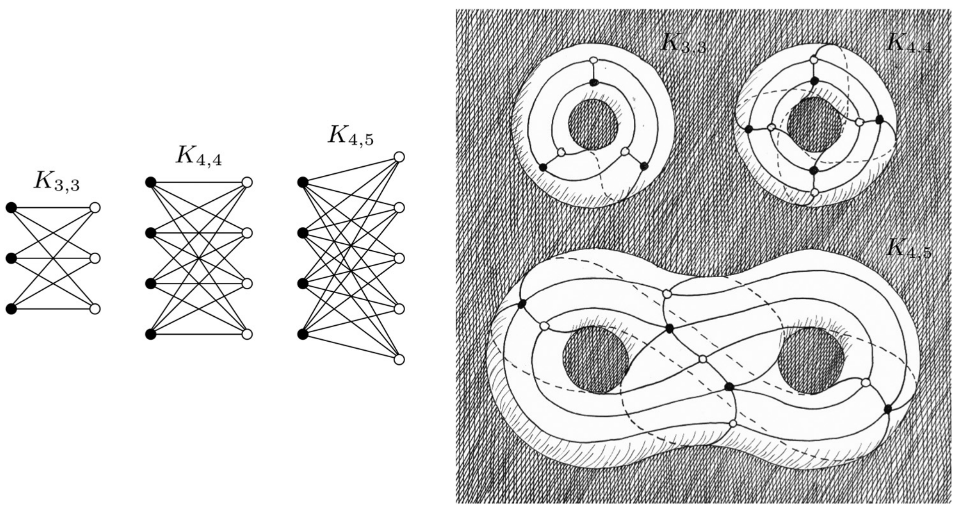


Figure 1. Left: The complete bipartite graph $K_{m,n}$ can be regarded as having m black vertices, n white vertices, and an edge joining any two vertices of different colors. Right: $K_{3,3}$, $K_{4,4}$, and $K_{4,5}$ on surfaces.

Figure 1 also shows $K_{4,4}$ and $K_{4,5}$, together with drawings of them on tori of genus 1 and 2, respectively. Indeed, Ringel [**16**] proved that for $2 \leq m \leq n$,

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil.$$

Thus $\gamma(K_{4,4}) = 1$ and $\gamma(K_{4,5}) = 2$, so our drawings in Figure 1 are optimal.

doi.org/10.1080/00029890.2020.1867472
MSC: Primary 57M15, Secondary 57M20; 05C10

Such genus formulas have been established for most well-known families of graphs. For instance, the difficult proof of the simple formula $\gamma(K_n) = \lceil (n-3)(n-4)/12 \rceil$ was instrumental in settling the Heawood map coloring conjecture [3, Chapter 7], [17].

For the n-dimensional hypercube graph Q_n, Ringel [15] and Beineke and Harary [2] used recursive arguments exploiting the hypercube's product structure to deduce

$$\gamma(Q_n) = 1 + (n-4)2^{n-3}. \tag{1}$$

Their proofs lead to generalizations like [9, 11–14, 19]. In contrast, our short, visual proof directly constructs the genus surface from the square faces of the n-cube and further shows that for odd n, the 2-skeleton of the n-cube is the union of $(n-1)/2$ copies of the genus surface, with no common faces, intersecting pairwise at Q_n.

Sections 2 and 3 define hypercubes, skeleta, and 2-cell embeddings of graphs in surfaces, and state Euler's formula for genus. Our proof of equation (1) is in Section 4 while Section 5 gives the bonus: a factorization of the 2-skeleton into genus surfaces.

2. HYPERCUBES. Let I denote the unit interval [0, 1] and let O denote its boundary $O := \partial I = \{0, 1\}$. (We use the notation O because it will be useful to regard an interval as being "active" [I] or "inactive" [O] in the manner described below.)

The **n-dimensional hypercube**, or **n-cube** is the polytope $H_n = I^n \subseteq \mathbb{R}^n$. Thus $H_n = \{(x_1, \ldots, x_n) \mid 0 \le x_i \le 1\} \subseteq \mathbb{R}^n$ is the intersection of the $2n$ half-spaces $x_i \ge 0$ and $x_i \le 1$, for $1 \le i \le n$. (See [20] for an introduction to polytopes.)

The 2^n **vertices** of H_n are the elements of O^n, which we identify with the binary strings of length n. (For example, $(1, 0, 1, 0)$ is 1010, etc.) The **edges** of H_n are the connected components of the products $O \times O \times \cdots \times I \times \cdots \times O$ having one active factor I and $n-1$ inactive factors O. Thus H_n has $n2^{n-1}$ edges, and each is a line segment joining two vertices that differ in exactly one coordinate (namely the active coordinate). The **faces** of H_n are the squares that are the connected components of

$$O \times \cdots \times I \times \cdots \times I \times \cdots \times O,$$

where two of the factors are I's and the rest are O's. Thus H_n has $\binom{n}{2}2^{n-2}$ faces, and the boundary of each face consists of four edges. Likewise H_n has $\binom{n}{3}2^{n-3}$ **3-faces**

$$O \times \cdots \times I \times \cdots \times I \times \cdots \times I \times \cdots \times O,$$

formed by choosing three positions for the I's. Each 3-face is a 3-cube whose boundary has six faces. In general, for each $0 \le k \le n$, the n-cube has $\binom{n}{k}2^{n-k}$ **k-faces** formed by choosing k active factors. Each k-face is a k-cube. (For brevity we call the 0-faces, 1-faces, and 2-faces *vertices, edges, and faces*, respectively, as stated above.)

The $(n-1)$-faces are called the **facets** of H_n. For each $1 \le k \le n$, there are two *opposite* facets that are the two components of $I \times I \times \cdots \times O \times \cdots \times I$, where the sole $O = \{0, 1\}$ is the kth factor. The reader should verify that any two nonopposite facets intersect in an $(n-2)$-face. Collectively the facets form the boundary of H_n.

The **k-skeleton** of H_n is the union of all its k-faces, so the 2-skeleton is the union of all the (square) faces. The 1-skeleton is the **hypercube graph**, denoted Q_n. Its vertices are the n-digit binary strings, and an edge connects any two vertices that differ in exactly one position. Figure 2 shows Q_2, Q_3, and Q_4.

The hypercube graph Q_n of any dimension is **bipartite**, that is, its vertices can be partitioned into two sets (say, black and white) such that each edge joins a black vertex

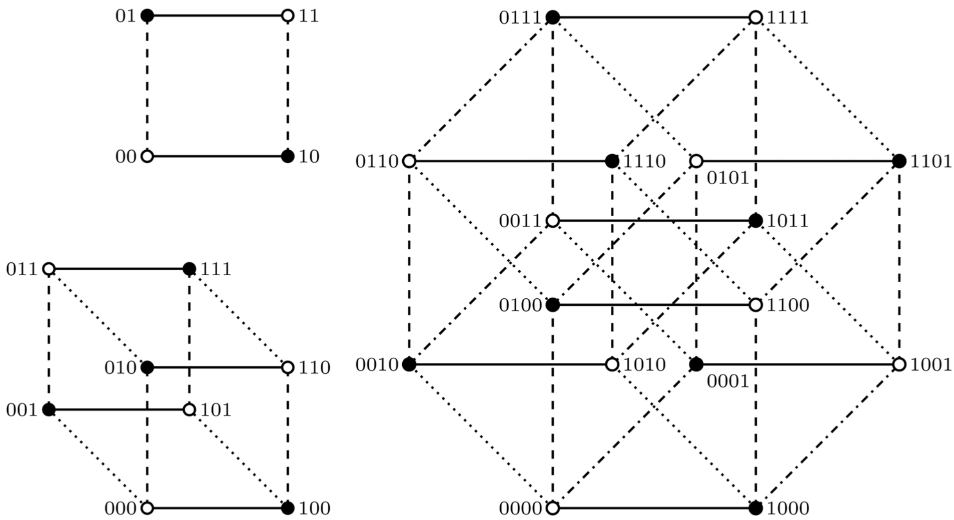


Figure 2. The 2-, 3-, and 4-dimensional cubes.

to a white vertex. (The black vertices are those binary strings with an odd number of 1's, while the white vertices have an even number of 1's, as in Figure 2.)

Figure 2 illustrates another nice feature of hypercubes: H_n can be edge-colored with colors $1, \ldots, n$, so that an edge whose endpoints differ in the kth coordinate gets color k. Note that each vertex is incident with exactly one edge of each color.

3. TWO-CELL EMBEDDINGS OF GRAPHS. Surfaces are mathematical objects that look locally like the plane, but which can differ radically from the plane globally. The Earth's surface is an example: it looks locally like a plane (at least on flat land) while globally it is not a plane at all, but a sphere. More exactly, a **surface** is a compact connected topological space in which each point has a neighborhood that is homeomorphic to an open disk. See introductory texts such as [**6**] and [**3**, Chapter 7] for development of the informal remarks we make here.

The **orientable** surfaces are the sphere, the torus, the 2-holed torus, and, in general, surfaces with g holes. (Nonorientable surfaces are those that one can cut a Möbius band out of, but they are not our focus here.) The number of holes in an orientable surface is called its **genus**. We denote the (unique!) surface of genus g by T_g. If T is an arbitrary orientable surface, its genus is denoted by $\gamma(T)$, so $\gamma(T_g) = g$. The sphere has genus 0, and is thus denoted as T_0. Figure 3 shows some examples.

As noted earlier, the **genus** $\gamma(G)$ of a graph G is the smallest integer g for which G can be drawn on T_g without crossed edges. Such a drawing of G on T_g is regarded as a continuous injection $G \to T_g$, and is called an **embedding** of G in T_g. An embedding of G in a surface of genus $\gamma(G)$ is called a **genus embedding** of G. Figure 1 shows genus embeddings of $K_{3,3}$, $K_{4,4}$, and $K_{4,5}$.

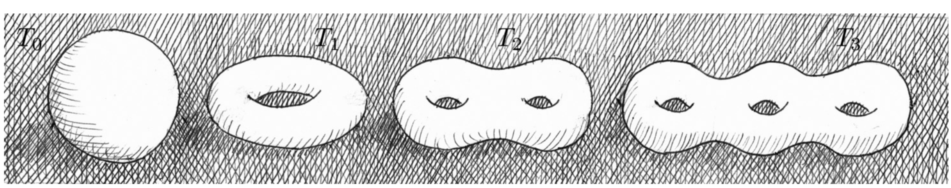


Figure 3. Examples of tori. The sphere T_0 (left) followed by T_1, T_2, and T_3.

An embedding $\varphi\colon G \to T$ divides T into **regions**, which are the connected components of $T \setminus \varphi(G)$. A **2-cell embedding** is one in which each region is homeomorphic to an open disk. The regions of a 2-cell embedding are called **faces**.

Every genus embedding is a 2-cell embedding [**3**, Theorem 7.2]. For example, the embedding of $K_{3,3}$ in Figure 1 has three faces: two squares and one octagon. The embedding of $K_{4,4}$ has eight faces, each a square.

Euler's formula [**3**, Theorem 7.1] implies that if a 2-cell embedding of a connected graph in a closed orientable surface T has v vertices, e edges, and f faces, then

$$\gamma(T) = \frac{2 - v + e - f}{2}. \tag{2}$$

Checking this on the embedding of $K_{4,4}$ in Figure 1 we get $1 = \frac{1}{2}(2 - 8 + 16 - 8)$. For $K_{4,5}$ in Figure 1, we get $2 = \frac{1}{2}(2 - 9 + 20 - f)$, so there are $f = 9$ faces.

If a 2-cell embedding of a connected graph with v vertices and e edges has f faces $F_1, \ldots, F_f$, and each F_i is a p_i-gon, then because each edge is on exactly two faces, we get $2e = p_1 + \cdots + p_f$. If the graph is bipartite, then it has no triangles, so $2e \geq 4f$. Then equation (2) yields a lemma [**3**, Corollary 7.6].

Lemma 1. *If G has v vertices, e edges, and is bipartite, then* $\gamma(G) \geq \dfrac{1}{4}(4 - 2v + e)$.

4. GENUS EMBEDDINGS OF HYPERCUBES.

We are ready for our theorem.

Theorem 1. *The genus of the hypercube graph Q_n is* $\gamma(Q_n) = 1 + (n-4)2^{n-3}$.

Proof. Color the edges of H_n with colors $1, 2, \ldots, n$, so that any edge joining vertices that differ in the kth coordinate is given the color k. Any face of H_n whose edges are colored k and ℓ is then "bicolored" by the pair $k\ell$. Assemble the collection $\mathcal{F}$ of all faces of H_n having one of the bicolors $12,\ 23,\ 34, \ldots,\ (n{-}1)n,\ n1$. For any one of these n bicolors, H_n has 2^{n-2} faces of that particular bicolor, so $|\mathcal{F}| = n2^{n-2}$. Any edge e of Q_n belongs to exactly two faces in $\mathcal{F}$: if e has color k, then these two faces have bicolors $(k{-}1)k$ and $k(k{+}1)$ (addition modulo n). Thus n faces of $\mathcal{F}$ are arranged cyclicly around each vertex, in the manner described by Figure 4. It follows that the faces $\mathcal{F}$ form a surface T in the 2-skeleton of H_n, with Q_n embedded in it. This surface has $n2^{n-2}$ square regions, and it is connected because Q_n is connected.

Assume for the moment that this surface is orientable. By the genus formula (2),

$$\gamma(T) = \frac{2 - v + e - f}{2} = \frac{2 - 2^n + n2^{n-1} - n2^{n-2}}{2} = 1 + (n-4)2^{n-3}.$$

Thus we have embedded Q_n in a surface of genus $1 + (n-4)2^{n-3}$. Is this the lowest genus possible? Lemma 1 says yes: $\gamma(Q_n) \geq \frac{1}{4}(4 - 2 \cdot 2^n + n2^{n-1}) = \gamma(T)$.

To finish the proof we must verify that T is orientable. There is indeed something to prove here, for when $n \geq 4$, the 2-skeleton of H_n contains Möbius strips (Figure 5, left). We must verify that none exist in T. To do this, note that each face of T has a local orientation given by the "right-hand rule" at either one of its black vertices: place your right hand at a black vertex with fingers pointing from edge color i to edge color $i{+}1$. Your thumb points in an "up" direction for this square, and your other fingers indicate a counterclockwise orientation for the square. (At white vertices, the thumb

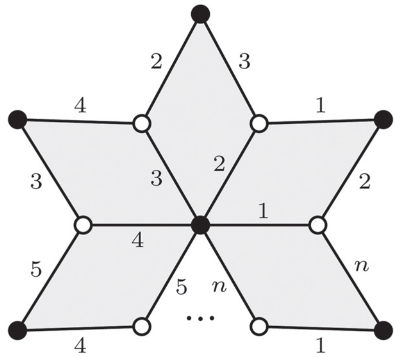
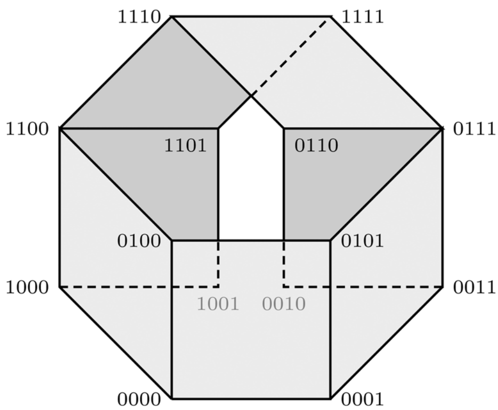


Figure 4. The squares in $\mathcal{F}$ that surround a vertex.

points "down.") Figure 5 shows that this orientation is preserved as we move from square to adjacent square on T. It follows that T is orientable. ■



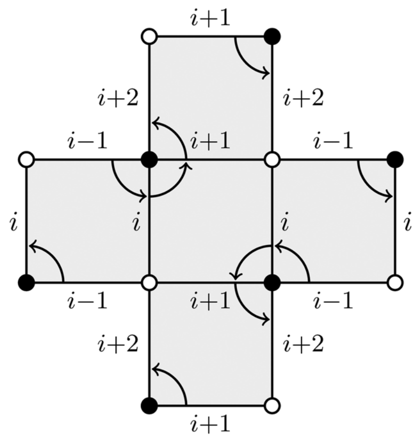


Figure 5. Left: A Möbius strip in H_4. Right: Squares in T are oriented by the right-hand rule at black vertices. Squares bicolored $(i-1)i$, $i(i+1)$, and $(i+1)(i+2)$ are shown.

Let's carry out the construction of this proof for Q_3, Q_4, and Q_5.

First, Q_3. Color the edges of Q_3 solid, dashed, and dotted, as in Figure 2. By our construction, we form a surface T by including the solid/dashed faces, the dashed/dotted faces, and dotted/solid faces of H_3. These are in fact all the faces of H_3, and we get the six faces shown in the upper left of Figure 6. They form the surface of the 3-cube, which is topologically equivalent to the sphere T_0.

Next, consider Q_4, with edges colored as in Figure 2. Our construction dictates that we form a surface T by including the solid/dashed faces, the dashed/dotted faces, the dotted/dash-dotted faces, and the dash-dotted/solid faces of H_4. There are sixteen such faces. We see them on the bottom left of Figure 6. The resulting surface is T_1, the torus. This surface does not include the solid/dotted and dashed/dash-dotted faces of H_4, but we clearly see their perimeters because their edges belong to the included faces. In walking through the hole of the torus, one walks through all four solid/dotted faces and sees the perimeters of all four dashed/dash-dotted faces, which are inside the torus. The representation of H_4 in Figure 2 has a long history. According to Robbin [18] such

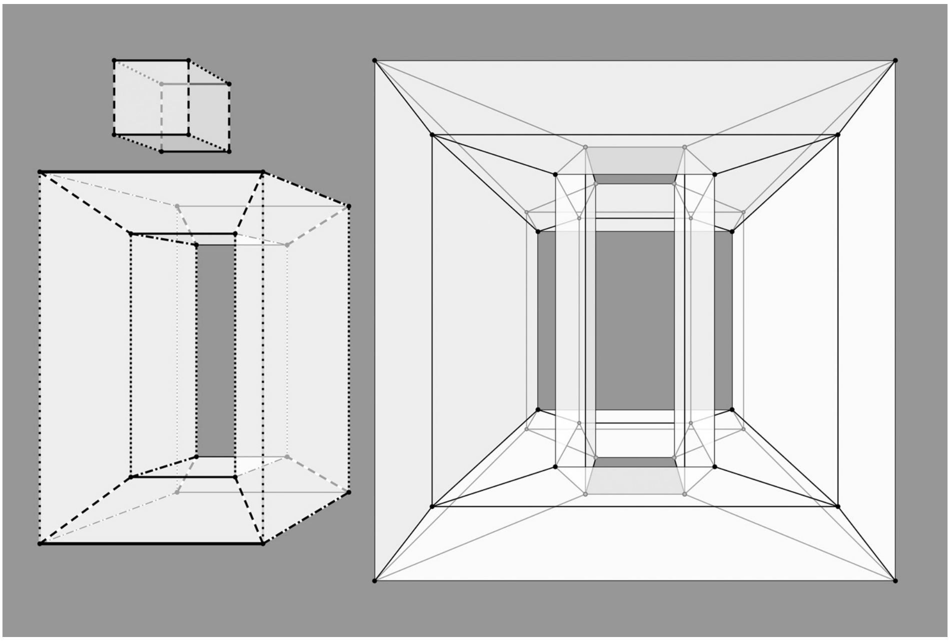


Figure 6. Genus embeddings of Q_3, Q_4, and Q_5.

a perspective view of H_4 (which is now standard) originated with V. Schlegel (1843–1905).

Let's now apply our construction to get a genus embedding of Q_5. We will not assign specific colors to the edges so as to avoid visual clutter in the image. Say the five edge colors of Q_5 are 1, 2, 3, 4, and 5. We thus include the 5-cube faces bicolored 12, 23, 34, 45, and 51 to obtain the embedding of Q_5 in T_5, shown in Figure 6.

This embedding does not include the 5-cube faces bicolored 13, 35, 52, 24, and 41. Thus the embedding uses exactly half the faces of the 5-cube. One can walk through this model and see all the edges and half the faces of H_5, without any intersections. The other half of the faces of the 5-cube form a surface isometric to this one. (See Section 5.) The missing faces are clearly visible because their perimeters are edges of faces that do belong to the embedding.

One nice feature of these embeddings is that they aid greatly in the visualization of hypercubes. One can, for example, easily pick out all ten facets of Q_5, and see how they fit together. To highlight this, we offer a few exercises related to our model of H_5 in T_5 (Figure 6).

Exercise 1: Locate all 80 faces (squares) of H_5 in this model.

Exercise 2: Identify all ten facets of H_5. (The facets are 4-cubes.)

Exercise 3: The 5-cube has 40 3-faces (each one a 3-cube). Find them (or at least some of them). For each one, locate the two 4-cube facets that share it.

Exercise 4: For an arbitrary vertex, find five 4-cube facets that share this vertex.

5. PARALLEL GENUS EMBEDDINGS.

In [2] and [15], the approach to cube embedding is *extrinsic*; the authors describe a recursive procedure that hooks together two lower-genus surfaces by a family of connecting "tubes." In contrast, our *intrinsic* method has the interesting consequence that the entire 2-skeleton can be decomposed into copies of the genus surface.

Recall that a cycle Z in a graph is a **Hamiltonian cycle** if Z contains all vertices of the graph. The **complete graph** K_n is the graph with vertex set $\{1, 2, \ldots, n\}$ and with an edge joining each pair of distinct vertices. Any ordering $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ gives rise to a Hamiltonian cycle $Z = i_1 i_2 \ldots i_n$ in K_n whose n edges are $\{i_k i_{k+1} \mid 1 \leq k \leq n\}$ (arithmetic modulo n).

Arguing as in the proof of Theorem 1, if $Z = i_1 i_2 \ldots i_n$ is *any* Hamiltonian cycle in K_n, then the union of the faces of H_n that are bicolored $i_k i_{k+1}$ ($1 \leq k \leq n$), addition mod n, is a surface $T(Z)$ that is a genus embedding for $Q_n \subseteq T(Z)$. In fact, if σ is the permutation $k \mapsto i_k$, $1 \leq k \leq n$, then σ induces a permutation of the standard unit vectors in $\mathbb{R}^n$ which carries T isometrically onto $T(Z)$ sending vertices to vertices, edges to edges, and faces to faces. If $Z' = i'_1 i'_2 \ldots i'_n$ is another Hamiltonian cycle in K_n and if Z and Z' share no edge, then $T(Z) \cap T(Z') = Q_n$.

We call a family of surfaces $T(Z_1), T(Z_2), \ldots, T(Z_s)$ a **parallel family** if every face (square) of H_n belongs to exactly one of the $T(Z_k)$. A collection of Hamiltonian cycles $Z_1, Z_2, \ldots, Z_s$ of K_n is a **Hamiltonian decomposition** of K_n if each edge belongs to exactly one of the cycles.

It has long been known that for odd n, K_n has a Hamiltonian decomposition into $(n-1)/2$ Hamiltonian cycles. (This was a problem in recreational mathematics, asking whether one could find seating arrangements around a round table that gave each pair of people a unique side-by-side appearance. See [**1**].) We get a nice consequence for the hypercube.

Theorem 2. *For odd $n \geq 3$, each Hamiltonian decomposition of K_n gives rise to a parallel family of $(n-1)/2$ genus embeddings of Q_n. That is, the 2-skeleton of H_n can be decomposed into face-disjoint isometric copies of genus embeddings of Q_n.*

Proof. Let $s = (n-1)/2$. Take a Hamiltonian decomposition $Z_1, \ldots, Z_s$ of K_n. The genus embeddings $T(Z_1), \ldots, T(Z_s)$ of Q_n intersect pairwise in Q_n. From the proof of Theorem 1, $T(Z_k)$ has $n2^{n-2}$ faces, so the s copies contain all $n2^{n-2}(n-1)/2 = \binom{n}{2}2^{n-2}$ faces of Q_n. ∎

After completing this article, we found that Das [**5**] also obtained Theorem 1 (but not Theorem 2) using a more general version of our approach, but avoiding the issue of nonorientability by induction on dimension. Das credits the idea of using the 2-skeleton to Coxeter's constructions of certain skew polyhedra [**4**]. Indeed, two of Coxeter's skew polyhedra coincide with our genus surfaces for Q_4 and Q_5, though Coxeter does not refer to graph genus.

In fact, the ideas above can be generalized in other directions. Any decomposition of K_n into edge-disjoint cycles, possible for $n \geq 3$ odd by a theorem of Euler [**8**, p. 64], yields a parallel family of surfaces for the 2-skeleton of H_n; see [**7**]. But the parallel family given by Theorem 2 has all surfaces pairwise isometric. Can this polytopal perspective be applied to other graph genus questions? What about the *nonorientable* genus of the n-cube [**10**]? It seems that many nice questions remain.

ACKNOWLEDGMENTS. We thank the referees for remarks that led to significant improvements. RHH is supported by Simons Foundation Collaboration Grant for Mathematicians 523748.

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