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Sphere decompositions of hypercubes*

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Abstract

For $d \equiv 1$ or $3 \pmod 6$, the 2-skeleton of the d-dimensional hypercube is decomposed into the union of pairwise face-disjoint isomorphic 2-complexes, each a topological sphere. If $d = 5^n$, then such a decomposition can be achieved, but with non-isomorphic spheres.

Keywords: Face-disjoint union of spheres, combinatorial design, 2-skeleton of a cube.

Math. Subj. Class. (2020): 57M20, 57M15, 05C45

By Euler's theorem [9, Prop. 1.2.27], any graph (1-complex) with all vertices of even degrees is an edge-disjoint union of cycles. We say a 2-complex is **even** if every edge lies in a positive even number of (2-dimensional) faces. Is every even 2-complex a face-disjoint union of "*2-dimensional cycles*"? (A 2-complex X is a **face-disjoint** union of 2-complexes $X_1, \dots, X_n$ if $X = \bigcup_{i=1}^n X_i$ and each face of X is a face of exactly one X_i.)

There are (at least) two natural choices for a 2-dimensional interpretation of cycle – *sphere* or *manifold*. As even complexes include surfaces like the torus, one cannot always decompose them into face-disjoint spheres. But we show below that sphere decompositions do exist in more than two-thirds of the odd-dimensional hypercubes. For $d \equiv 1$ or 3 (mod 6), we can decompose the 2-skeleton Q_d^2 of the d-dimensional hypercube Q_d into face-disjoint copies of ∂Q_3, the boundary of a 3-cube. That is, Q_d^2 is **factored by** ∂Q_3.

In [6], when d is odd (so the 2-skeleton is even), Q_d^2 is decomposed into a face-disjoint union of tori and 3-cube boundaries. In [4] we showed that the 2-skeleton of any d-dimensional Platonic polytope is a face-disjoint union of surfaces if the 2-skeleton is even. Except for the hypercubes, all such decompositions were decompositions into spheres. (A polytope is **Platonic** if it is maximally symmetric. In dimension greater than four, the Platonic polytopes are just the cubes, simplexes, and hyperoctahedra.)

For which odd d is the 2-skeleton of the d-cube decomposable into spheres? For which d can the decomposition be a factorization? We address these questions below.

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Throughout this paper I denotes the interval $[0, 1]$ and O its boundary $O = \{0, 1\}$. (We use the non-standard notation O for ∂I because it will be convenient to think of an interval as being "active" (I) or "inactive" (O) in the manner indicated below.) We regard the d-cube as $Q_d = I^d \subseteq \mathbb{R}^d$. Thus the 2^d **vertices** of Q_d are the elements of O^d, which we identify with the binary strings of length d. An **edge** of Q_d is a line segment joining two vertices that differ in exactly one position (i.e., coordinate). Selecting a coordinate i from 1 to d, there are 2^{d-1} edges among the connected components of $O \times O \times \cdots \times I \times \cdots \times O$, where the sole ("active") factor I occurs in the ith position. Thus Q_d has $d2^{d-1}$ edges. The **faces** of Q_d are the squares that are the connected components of

$$O \times \cdots \times I \times \cdots \times I \times \cdots \times O,$$

where exactly two of the factors are I's and the rest are O's. Thus Q_d has $\binom{d}{2}2^{d-2}$ faces, and the boundary of each face consists of four edges. Likewise Q_d has $\binom{d}{3}2^{d-3}$ **3-facets**

$$O \times \cdots \times I \times \cdots \times I \times \cdots \times I \times \cdots \times O,$$

formed by selecting three positions for the I's. Each 3-facet is a 3-cube whose boundary consists of six faces. Similarly, Q_d has $\binom{d}{k}2^{d-k}$ k-facets for each $0 \leq k \leq d$, and each k-facet is a k-cube. The 2**-skeleton**, Q_d^2, of Q_d is the union of all of its faces.

Notice that each edge of Q_d belongs to $d-1$ faces, so the 2-skeleton is even if and only if d is odd. Hence Q_d^2 has no sphere decomposition if d is even.

1 Sphere decompositions in dimensions 1 and 3 (mod 6)

Here we show that if $d = 3, 7, 9, 13, 15, 19, 21, \ldots$, that is, if $d \equiv 1$ or $3 \pmod 6$, then the 2-skeleton of Q_d can be decomposed into a face-disjoint union of boundaries of 3-cubes.

We use combinatorial designs [1], [8, pp. 96–100]. Let $[d] := \{1, \ldots, d\}$. A **k-design** $S(k, d)$ on $[d]$ is a family of k-subsets of $[d]$ (called **blocks**) such that each 2-subset of $[d]$ is contained in a unique block. Though 3-designs are called *Steiner triple systems*, it was Kirkman [7] who proved that they exist if and only if $d \equiv 1$ or 3 (mod 6). Conditions that are algebraically necessary turned out to be combinatorially sufficient.

Before describing our general construction we illustrate it for Q_7. We will decompose the 2-skeleton of Q_7 into 112 pairwise face-disjoint 3-cube boundaries. The first step is to realize a Steiner triple system $S(3, 7)$. Label the vertices of a 7-gon with the integers 1 through 7, as in in Figure 1. The shaded triangle on the left has vertices 1, 2 and 4, and any two of them are a distance of 1, 2 or 3 apart along the 7-gon. Rotating the triangle in multiples of $2\pi/7$ yields seven triangles, whose respective vertex sets are tallied below them. These are the blocks of $S(3, 7)$ because any two vertices on the 7-gon are at distance 1, 2, or 3, and therefore they are vertices of exactly one of the triangles.

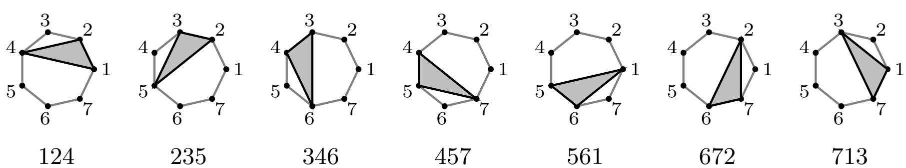

Figure 1: Construction of a Steiner triple system $S(3, 7)$.

Each block of $S(3,7)$ corresponds to one of seven classes of 3-cubes in Q_7 indicated in Table 1, where an integer i belongs to the block if and only if the product is active in the ith factor. Notice that permuting the factors in a class cyclicly yields the subsequent class.

124	$I \times I \times O \times I \times O \times O \times O$
235	$O \times I \times I \times O \times I \times O \times O$
346	$O \times O \times I \times I \times O \times I \times O$
457	$O \times O \times O \times I \times I \times O \times I$
561	$I \times O \times O \times O \times I \times I \times O$
672	$O \times I \times O \times O \times O \times I \times I$
713	$I \times O \times I \times O \times O \times O \times I$

Table 1: The seven classes of 3-cubes in Q_7.

As $O = \{0,1\}$, each of the seven classes contains 16 disjoint 3-cubes, for a total of 112 3-cubes. Notice that any two cubes from the same class have empty intersection. Further, two 3-cubes from different classes are either disjoint or they intersect at an edge because by construction they have exactly one I as a common factor. We have accounted for $6 \cdot 112 = 672$ faces of Q_7, which has indeed $\binom{7}{2}2^5 = 672$ faces. We therefore have a decomposition of its 2-skeleton into pairwise face-disjoint boundaries of 3-cubes.

To visualize this, let $P : \mathbb{R}^7 \to \mathbb{R}^2$ be the projection sending the standard basis elements $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_7$ to the vertices of a regular 7-gon, cyclically, as in Figure 1. Figure 2 (left) shows the projection P of the 16 disjoint 3-cubes in the class $I \times I \times O \times I \times O \times O \times O$ (shown bold in the figure, with other edges of Q_7 gray). There is much overlap in this figure. The right of Figure 2 shows the same projection, but with the vectors $P(\mathbf{e}_1)$, $P(\mathbf{e}_2)$ and $P(\mathbf{e}_4)$ scaled by a factor of about 0.2 in order to separate the 3-cubes. Observe that rotating Figure 2 (left) by $2\pi/7$ brings the cubes $I \times I \times O \times I \times O \times O \times O$ to the cubes $O \times I \times I \times O \times I \times O \times O$, etc.

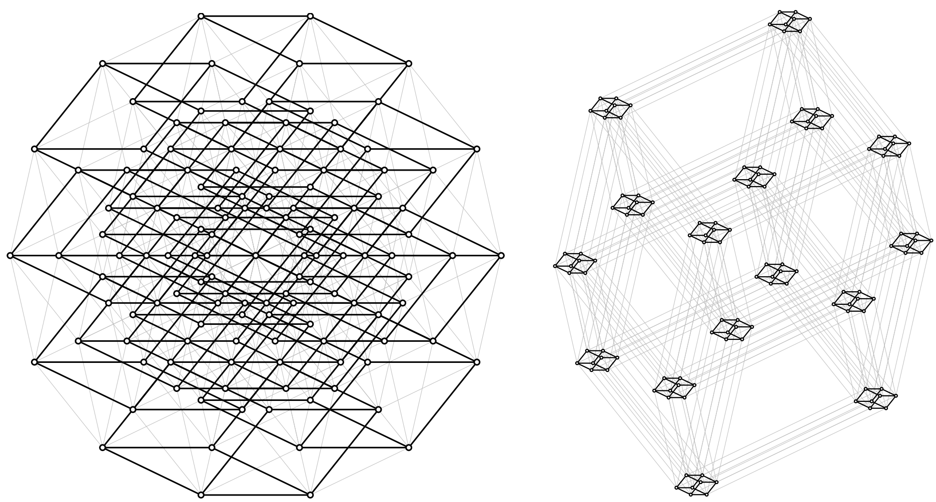

Figure 2: Two views of the sixteen 3-cubes $I \times I \times O \times I \times O \times O \times O$ (bold lines) in Q_7.

Now that we have illustrated our construction, we can prove the general result.

Theorem 1.1. *The* 2*-skeleton of* Q_d *can be decomposed into a pairwise face-disjoint union of 3-cube boundaries if and only if* $d \equiv 1$ *or* $3 \pmod 6$.

Proof. Let $d \equiv 1$ or $3 \pmod 6$ and let $S(3,d)$ be a 3-design. As $[d]$ has $\binom{d}{2}$ pairs and each block of $S(3,d)$ contains $\binom{3}{2} = 3$ pairs, the number of blocks is $\frac{1}{3}\binom{d}{2} = \frac{d(d-1)}{6}$. For each block $\{i,j,k\}$ of $S(3,d)$, construct a class of 3-cubes

$$O \times \cdots \times I \times \cdots \times I \times \cdots \times I \times \cdots \times O,$$

where there is an I precisely in the ith, jth and kth factors. Such a class consists of 2^{d-3} disjoint 3-cubes. By construction, the intersection of any two 3-cubes from different classes corresponding to blocks $\{i,j,k\}$ and $\{i',j',k'\}$ is either empty, or a vertex, or an edge. Indeed, the intersection cannot be a face in any $O \times \cdots \times I \times \cdots \times I \times \cdots \times O$ because this would mean that some pair belongs to both $\{i,j,k\}$ and $\{i',j',k'\}$. Thus these 3-cubes are pairwise face-disjoint. The cubes in the $\frac{d(d-1)}{6}$ classes thus account for $6\frac{d(d-1)}{6}2^{d-3} = \binom{d}{2}2^{d-2}$ faces of Q_d, which is all of the faces of Q_d. We have thus decomposed the 2-skeleton of Q_d into a pairwise face-disjoint union of boundaries of 3-cubes.

Conversely suppose that $d \not\equiv 1$ or $3 \pmod 6$. If d is even, then Q_d^2 is not even, so it does not have a sphere decomposition. Thus assume d is odd, in which case $d \equiv 5 \pmod 6$. An easy computation shows that, in this case, the number of faces in Q_d^2 is not a multiple of 6. Hence Q_d^2 cannot be decomposed as a pairwise face-disjoint union of 3-cubes. □

Theorem 1.1 does not cover the cases $d = 5, 11, 17, 23, \ldots$, where $d \equiv 5 \pmod 6$. We do not know if all such such Q_d^2 have sphere decompositions. In the next section we find sphere decompositions when $d = 5^n$. However, these decompositions are not factorizations as they involve non-isomorphic complexes.

2 A sphere decomposition of the 5-cube

We now show that there is a sphere decomposition for Q_5^2, which is the smallest case not covered by a Steiner triple system. In fact, we will get somewhat more. Theorem 2.1 below guarantees sphere decompositions of Q_d^2 exist for arbitrarily large $d \equiv 5 \pmod 6$.

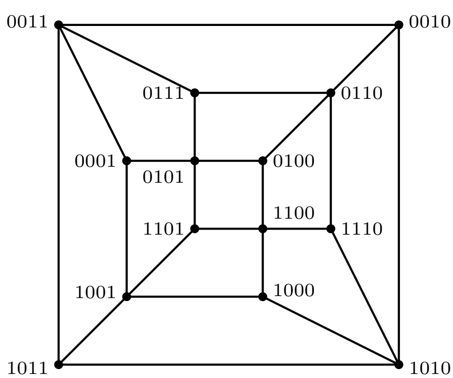

Figure 3: The 2-skeleton of the 4-cube, minus the vertices 0000 and 1111, is a sphere S.

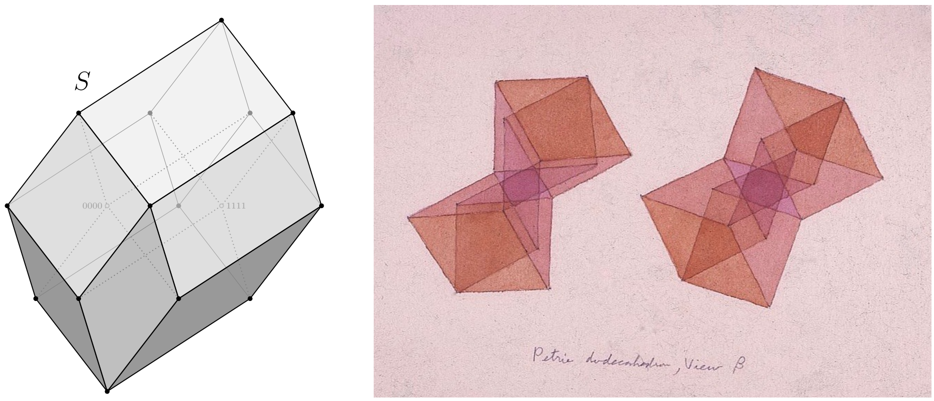

Figure 4: The rhombic dodecahedron obtained by deleting opposite vertices of Q_4^2. The watercolor (right) by David W. Brisson (1977) is a *hypersterogram* [2] showing two views differing in two degrees of parallax. Used with permission of Harriet and Erik Brisson.

Theorem 2.1. *If $d = 5^n$, then the 2-skeleton of Q_d is a face-disjoint union of spheres.*

Proof. We first treat the case $d = 5$. The case $d = 5^n$ will follow from design theory.

Our plan is to realize the 2-skeleton of Q_4 as a face-disjoint union of a sphere S and six disks $D_1, \ldots, D_6$ with edge-disjoint boundaries, then show that the 2-skeleton of Q_5 is the face-disjoint union of the eight spheres $S \times \{0\}$, $S \times \{1\}$, $\partial(D_1 \times [0,1]), \cdots, \partial(D_6 \times [0,1])$.

Let $S = Q_4^2 - \{0000, 1111\}$ be Q_4^2 with the antipodal vertices 0000 and 1111 removed (and with them all the edges and faces incident with them). We thus have removed two vertices, eight edges and 12 faces. What remains is a sphere S with 12 square faces. It is shown in Figure 3 embedded in the punctured sphere (plane). We note in passing that sphere S is a rhombic dodecahedron, which can be embedded in $\mathbb{R}^3$ with 12 congruent rhombic faces. (See Figure 4.)

The sphere S accounts for 12 of the 4-cube's 24 faces. The 12 missing squares are all incident with one or the other of the removed vertices 0000 and 1111. Figure 5 shows eight of these missing squares. Four of them form a disk D_1 centered at 0000 and the other four make a disk D_2 centered at 1111. These disks are pairwise face-disjoint, and their boundaries are pairwise edge-disjoint. And none of their faces are faces of S, because each face of D_1 and D_2 contains either the vertex 0000 or 1111, and neither of these vertices is in S.

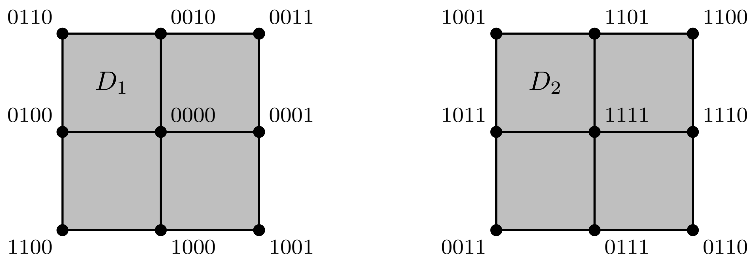

Figure 5: The disks D_1 and D_2 centered at 0000 and 1111, respectively.

So far we have accounted for 20 squares of Q_4^2, 12 of them in S, four in D_1, and four in D_2. There are just four squares in Q_4^2 that are unaccounted for. They are not hard to find, because 0000 and 1111 are each contained in *six* squares of Q_4^2 and Figure 5 shows only four squares at 0000 and 1111. Thus the four missing squares are incident with 0000 or 1111. They are shown in Figure 6, superimposed on the drawings from Figure 5. Call these four squares disks D_3, D_4, D_5 and D_6.

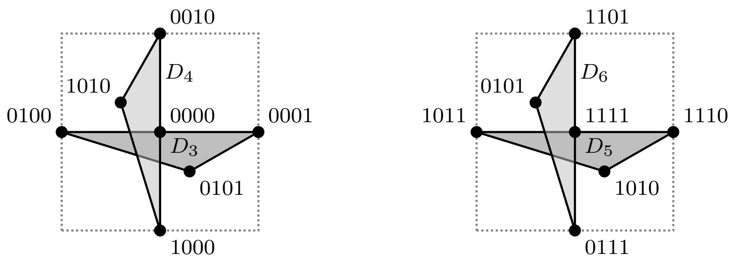

Figure 6: The disks D_3, D_4, D_5 and D_6.

Note that the sphere S and disks $D_1, D_2, \dots D_6$ are pairwise face-disjoint and account for all squares of Q_4^2. Further the boundaries of the disks are pairwise edge-disjoint. We now have eight spheres in Q_5^2: $S \times \{0\}$, $S \times \{1\}$, $\partial(D_1 \times [0,1]), \cdots, \partial(D_6 \times [0,1])$. By construction they are face-disjoint. (See Figure 7.) Moreover the total number of squares used is $12 + 12 + 16 + 16 + 6 + 6 + 6 + 6 = 80$, so we have used all the squares in Q_5^2.

We have now decomposed the 2-skeleton of Q_5 into a pairwise face-disjoint union of spheres, two of which are rhombic dodecahedrons, two of which have the structure shown in Figure 7 (left), and four of which are the boundaries of a 3-cube, as in Figure 7 (right).

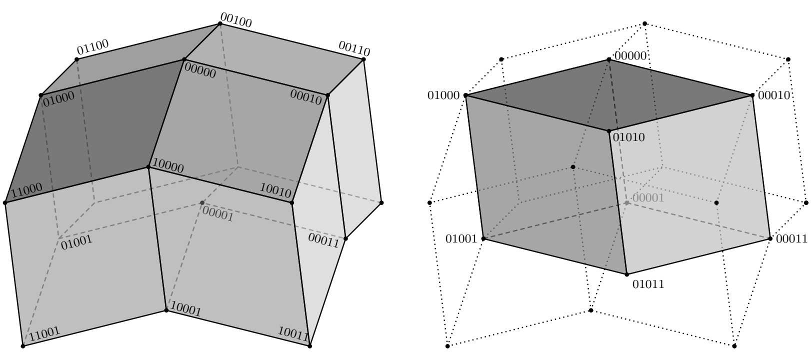

Figure 7: The spheres $\partial(D_1 \times I)$ (left) and $\partial(D_3 \times I)$ (right) intersect at the hexagon 00000–00010–00011–00001–01001–01000–00000. Our decomposition of of Q_5 uses two spheres of the type on the left, four of the type on the right, and two rhombic dodecahedra.

Having obtained a sphere decomposition of Q_5^2, we get a generalization. Consider the finite field $\mathbb{F}_5$ consisting of the integers modulo 5. The vector space $\mathbb{F}_5^n$ then consists of 5^n elements, or **points**, and each 1-dimensional subspace $V = \{\lambda \mathbf{v} \mid \lambda \in \mathbb{F}_5\}$ consists of five points. A **line** L is a translate $L = \{\mathbf{w} + \lambda \mathbf{v} \mid \lambda \in \mathbb{F}_5\}$ of a 1-dimensional subspace. We can realize $S(5, 5^n)$ by letting the blocks be the lines in $\mathbb{F}_5^n$. (Each line consists of 5 of the 5^n points in $\mathbb{F}_5^n$, and any two points in $\mathbb{F}_5^n$ lie on a unique line.) From each block we can extract 10 pairs of points, so the total number of blocks is $\frac{1}{10}\binom{5^n}{2} = \frac{5^{n-1}(5^n-1)}{4}$. Using the development from Section 1, it follows that the 2-skeleton of Q_{5^n} is the face-disjoint union of $\frac{5^{n-1}(5^n-1)}{4}2^{5^n-5}$ 5-cubes, each of which is decomposable into a pairwise face-disjoint union of spheres. We can thus decompose the 2-skeleton of Q_{5^n} into a pairwise face-disjoint union of spheres. Indeed, the total number of faces used in this decomposition is $80\frac{5^{n-1}(5^n-1)}{4}2^{5^n-5} = \frac{5^n(5^n-1)}{2}2^{5^n-2} = \binom{5^n}{2}2^{5^n-2}$, the number of faces of $Q_{5^n}^2$. $\square$

Notice that $5^n \equiv 5 \pmod 6$ if and only if n is odd, so Theorem 2.1 yields a new class of hypercubes with sphere decompositions that is not covered by Theorem 1.1.

3 Discussion

Design theory applies to additional cases where $d \equiv 5 \pmod 6$ by using the technique of the previous section. Suppose one has a sphere decomposition of some Q_k^2 and there is a k-design on $[d]$. Then there is a sphere decomposition for Q_d^2. We illustrate this for $k = 5$.

In [5, Thm. 2], Hanani showed that a 5-design exists if and only if $d \equiv 1$ or 5 (mod 20). So for $d = 41, 65,$ etc., any $S(5, d)$ and any sphere decomposition of Q_5^2 can be combined to construct a sphere decomposition of Q_d^2 for some $d \neq 5^n$.

We conjecture that sphere decompositions exist for Q_d^2 for all odd d, but that spherical factorizations exist if and only if $d \equiv 1$ or 3 (mod 6).

Note that cyclical configurations of points and lines were constructed by Grünbaum through a similar use of Steiner triple systems. See [3, pp. 253 and 325].

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References

[1] E. F. Assmus, Jr. and J. D. Key, *Designs and their codes*, volume 103 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 1992, doi:10.1017/cbo9781316529836.

[2] D. Brisson, *Hypergraphics: Visualizing Complex Relationships in Art, Science and Technology*, American association for the advancement of science. Selected symposia Series 24, Westview, 1978, https://books.google.com/books?id=J6yOzQEACAAJ.

[3] B. Grünbaum, *Configurations of points and lines*, volume 103 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2009, doi:10.1090/gsm/103.

[4] R. Hammack and P. C. Kainen, On 2-skeleta of Platonic polytopes, *Bull. Hellenic Math. Soc.* **62** (2018), 94–102, http://bulletin.math.uoc.gr/bulletin/work/autolist.php?decade=2010.

[5] H. Hanani, On balanced incomplete block designs with blocks having five elements, *J. Combinatorial Theory Ser. A* **12** (1972), 184–201, doi:10.1016/0097-3165(72)90035-0.

[6] P. C. Kainen, On 2-skeleta of hypercubes, *Art Disc. Appl. Math.* **3** (2020), #P2.06, 4p., doi:10.26493/2590-9770.1302.f4e.

[7] T. P. Kirkman, On a problem in combinations, *Cambridge and Dublin Mathematical Journal* **2** (1847), 191–204, http://resolver.sub.uni-goettingen.de/purl?PPN600493962_0002.

[8] H. J. Ryser, *Combinatorial mathematics*, The Carus Mathematical Monographs, No. 14, Published by The Mathematical Association of America; distributed by John Wiley and Sons, Inc., New York, 1963, doi:10.5948/upo9781614440147.

[9] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 2001, https://books.google.com/books?id=TuvuAAAAMAAJ.