Test One	Advanced Graph Theory MATH 656	March 5, 2020
Name: ______________________	R. Hammack	Score: ________

Directions: Choose any four questions. Each of your four chosen questions is 25 points, for a total of 100 points. If you do more than four questions, please clearly indicate which of the four you want to contribute toward your 100 points.

Solutions are given below for problems done or attempted by test-takers.

1. Prove: A graph G is m-colorable if and only if $\alpha(G \square K_m) \geq n(G)$. ($\alpha$ is the independence number.)

Proof. Suppose G is m-colorable, so $\chi(G) \leq m$. Using the fact that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$, we get

$$\chi(G \square K_m) = \max\{\chi(G), \chi(K_m)\} = m.$$

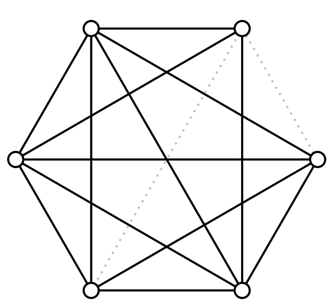
By a standard lower bound for the chromatic number, $\chi(G \square K_m) \geq \dfrac{n(G \square K_m)}{\alpha(G \square K_m)}$, and the above gives $m \geq \dfrac{n(G \square K_m)}{\alpha(G \square K_m)}$. But $n(G \square K_m) = n(G) \cdot n(K_m) = n(G) \cdot m$, so the above becomes $m \geq \dfrac{n(G) \cdot m}{\alpha(G \square K_m)}$, hence $\alpha(G \square K_m) \geq n(G)$.

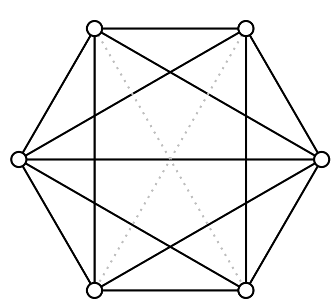
Conversely, suppose that $\alpha(G \square K_m) \geq n(G)$. Label the vertices of K_m as $1, 2, 3, \ldots, m$. Choose an independent set $X \subseteq V(G \square K_m) = V(G) \times \{1, 2, \ldots, m\}$ with $|X| = n(G)$. Observe that any two vertices $(x, i), (x, j) \in V(G \square K_m)$ with equal first coordinates are adjacent, so the $n(G)$ vertices of X have $n(G)$ distinct first coordinates. In other words, to any $x \in V(G)$, there is exactly one vertex $(x, k) \in X$.

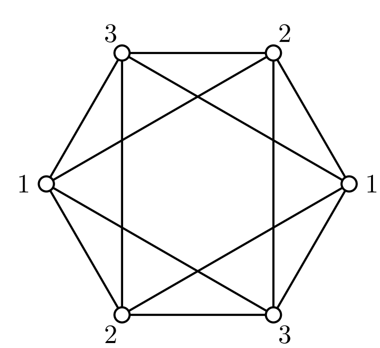
For each $k \in V(K_m)$, let $X_k = \{x \in V(G) \mid (x, k) \in X\} \subseteq X$. Notice that each X_k is an independent set in G: Indeed, if $x, y \in X_k$, then $(x, k), (y, k) \in X$, and $xy \in E(G)$ would force $(x, k)(y, k) \in E(G \square K_m)$, violating the independence of X. Also note that by their definition, the X_k have pairwise empty intersection (though it is possible that some of them are themselves empty). Further, $X = X_1 \cup X_2 \cup \cdots \cup X_m$. Give each vertex (if any) in X_k the color k. This properly colors G with at most m colors, so m is m-colorable. ■

2. (a) Prove that there is no simple graph with 6 vertices and 13 edges that has chromatic number 3.

Proof. The greatest number of edges that a 6-vertex simple graph could have is $\binom{6}{2} = 15$, in which case it is complete, so a 6-vertex graph G with 13 edges is just a complete graph with two edges removed. There are only two possibilities for such a graph: the removed edges are incident, or the removed edges are not incident. These cases are illustrated below, left and center. The graph on the left is K_6 with two incident edges removed; it contains a K_5, so it is not 3-chromatic. The middle graph is K_6 with two non-incident edges removed. It has a K_4, so it is not 3-chromatic. Thus there is no simple graph with 6 vertices and 13 edges that has chromatic number 3. ■







(b) Give an example of a simple graph with 6 vertices and 12 edges that has chromatic number 3.

See above, right. ■

4 Let G be a simple graph with n vertices. Recall that $\chi(G;k) = \sum_{r=1}^{n} p_r(G)k(k-1)(k-2)\cdots(k-r+1)$, where $p_r(G)$ is the number of partitions of $V(G)$ into r non-empty independent sets. Use this to prove that the coefficient of k^{n-1} in $\chi(G;k)$ is $-e(G)$.

Proof. Since $k(k-1)(k-2)\cdots(k-r+1)$ has degree r, the only terms of the sum that can contribute to k^{n-1} are $r = n$ and $r = n-1$. Clearly $p_n(G) = 1$. Further, $p_{n-1}(G) = \binom{n}{2} - e(G)$, because in any partition of $V(G)$ into $n-1$ independent sets, exactly one independent set has two elements, and all others have just one element, so the one with two elements must consist of two *non-adjacent* vertices. It follows that the number of partitions of $V(G)$ into $n-1$ non-empty independent sets is the number of edges in $\overline{G}$, which is $\binom{n}{2} - e(G)$. Thus we seek the coefficient of k^{n-1} in

$$p_n(G)k(k-1)(k-2)\cdots(k-n+1) + p_{n-1}(G)\,k(k-1)(k-2)\cdots(k-(n-1)+1),$$

which is

$$(k-0)(k-1)(k-2)\cdots(k-(n-1)) + \left[\binom{n}{2} - e(G)\right](k-0)(k-1)(k-2)\cdots(k-(n-2)).$$

The product $(k-0)(k-1)(k-2)\cdots(k-(n-2))$ on the right is monic of degree $n-1$, so the coefficient of k^{n-1} in the right-hand expression is $\binom{n}{2} - e(G)$.

Now let's look at the polynomial $(k-0)(k-1)(k-2)\cdots(k-(n-1)) = \prod_{i=0}^{n-1}(k-i)$ on the left, which has degree n. In its formal expansion, we pick up a k^{n-1} term by multiplying a $-i$ from one of the factors $(k-i)$ by $n-1$ k's from the remaining $n-1$ factors $(k-j)$. Therefore, the coefficient of k in $(k-0)(k-1)(k-2)\cdots(k-(n-1))$ is $-1-2-3-4-\cdots-(n-1) = -\binom{n}{2}$.

It follows that the coefficient of k^{n-1} in $\chi(G;k)$ is $-\binom{n}{2} + \left[\binom{n}{2} - e(G)\right] = -e(G)$. ■

5 Prove: If a graph G has c components, then $\chi(G;k) = k^c f(k)$, where $f(0) \neq 0$.

Proof. First we claim that if this is true for connected graphs then it is true for any graph. Suppose that if G is connected, then $\chi(G;k) = kf(k)$, where $f(0) \neq 0$. Now let G be a disconnected graph with components $G_1, G_2, \ldots, G_c$, so for each index i we have $\chi(G_i;k) = kf_i(k)$ for some polynomial f_i with $f_i(0) \neq 0$. The number of proper k-colorings of G is $\chi(G;k)$, which by the multiplication principle is $\prod_{i=1}^{c} \chi(G_i;k) = \prod_{i=1}^{c} kf_i(k) = k^c\big(\prod_{i=1}^{c} f_i(k)\big) = k^c\big(\prod_{i=1}^{c} f_i\big)(k)$. Thus $\chi(G;k) = k^c\big(\prod_{i=1}^{c} f_i\big)(k)$, and $\big(\prod_{i=1}^{c} f_i\big)(0) \neq 0$ because all the $f_i(0)$ are non-zero.

This shows that the result is true provided it is true for connected graphs, so to complete the proof we need to show that if G is connected, then $\chi(G;k) = kf(k)$ for some polynomial f with $f(0) \neq 0$. The following claim completes the proof.

Claim: If G is connected then $\chi(G;k) = kf(k)$, where $f(0) < 0$ if $n(G)$ is even, and $f(0) > 0$ if $n(G)$ is odd.

The proof of the claim is induction on $e(G)$.

Basis step: First suppose $e(G) = 0$. Then $G = K_1$, and $n(G)$ is odd, while $\chi(G;k) = kf(k)$, where f is the identity polynomial $f(k) = 1 = k^0$. Ineed $f(0) = 1 > 0$. Next suppose $e(G) = 1$. Then $G = K_2$, and $n(G)$ is even, while $\chi(G;k) = k(k-1) = kf(k)$, where $f(k) = (k-1)$. Indeed $f(0) = -1 < 0$.

Now let G have $e > 1$ edges, and assume the claim is true for any graph with fewer than e edges. If G is a tree, then $\chi(G) = k(k-1)^{n(G)-1}$. Here $f(k) = (k-1)^{n(G)-1}$, and indeed $f(0) = (0-1)^{n(G)-1} = (-1)^{n(G)-1}$ is negative for even $n(G)$ and positive for odd $n(G)$. If G is not a tree, then choose and edge $e \in E(G)$ that lies on a cycle, so that $G-e$ is connected. By chromatic recurrence and the inductive hypothesis, we get

$$\begin{aligned} \chi(G;k) &= \chi(G{-}e;k) - \chi(G \cdot e;k) \\ &= kf(k) - kg(k) \\ &= k\big(f(k) - g(k)\big), \end{aligned}$$

Where $f(0)$ is negative if $n(G{-}e)$ is even, and positive if $n(G{-}e)$ is odd; and also $g(0)$ is negative if $n(G \cdot e)$ is even, and positive if $n(G \cdot e)$ is odd.

If $n(G)$ is even, then $n(G{-}e)$ is even and $n(G \cdot e)$ is odd, and we get $\big(f(0) - g(0)\big) < 0$.
If $n(G)$ is odd, then $n(G{-}e)$ is odd and $n(G \cdot e)$ is even, and we get $\big(f(0) - g(0)\big) > 0$. ■

9 Prove that every simple planar graph with at least four vertices has at least four vertices of degree less than 6.

Proof. Let G be planar with at least four vertices. We need to show that there are four vertices of G that have degrees less than 6. Because adding edges cannot decrease vertex degrees, it suffices to prove that if G is a *maximal planar*, then at least four of its vertices have degrees less than 6.

Put $n = n(G)$. Since G is maximal planar, $e(G) = 3n - 6$. Let G have a_0 vertices of degree 0, a_1 vertices of degree 1, a_2 vertices of degree 2, a_3 vertices of degree 3, and so on, up to a_{n-1} vertices of degree $n-1$. Then

$$\begin{aligned} 2e(G) &= 0a_0 + 1a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + \cdots + (n-1)a_{n-1} \\ 2(3n-6) &= 0a_0 + 1a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + \cdots + (n-1)a_{n-1} \\ 6n - 12 &= 0a_0 + 1a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + \cdots + (n-1)a_{n-1} \end{aligned}$$

Because $n \geq 4$ and G is maximal planar, there are no vertices of degrees 0, 1, or 2, so $a_0 = a_1 = a_2 = 0$. Further, $n = a_3 + a_4 + \cdots + a_{n-1}$. Thus the above becomes

$$\begin{aligned} 6\Big(a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + a_{n-1}\Big) - 12 &= 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + \ldots + (n-1)a_{n-1} \\ 3a_3 + 2a_4 + a_5 &= 12 + a_7 + 2a_8 + 3a_9 + \cdots + (n-7)a_{n-1}. \end{aligned}$$

Consequently $3a_3 + 2a_4 + a_5 \geq 12$, which is impossible if $a_3 + a_4 + a_5 < 4$. Therefore $a_3 + a_4 + a_5 \geq 4$, which is to say that G has at least 4 vertices of degree 3, 5 or 5. ■