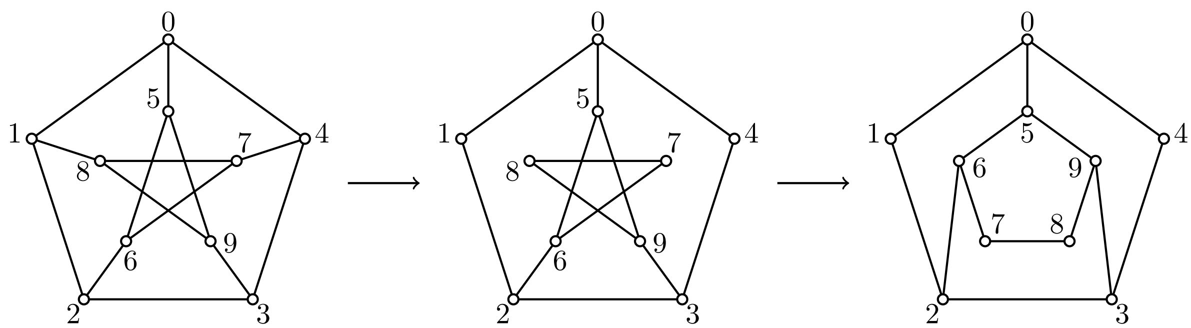
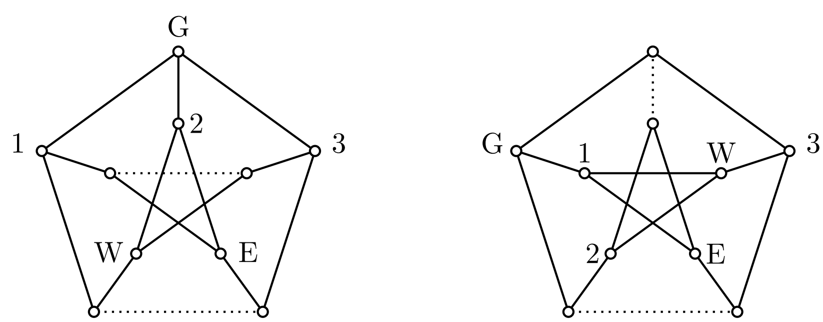
6.2.5 Determine the minimum number of edges that must be deleted from the Petersen graph to obtain a planar graph.

We claim that the number of edges is exactly 2. First, the number of edges is at least 2, because deletion of the two edges 18 and 74 in the following picture of the Petersen graph results in a planar graph, as illustrated.



But if we delete only one edge—wether it be on the outer pentagon, the inner pentagram star, or an edge connecting the two—we can still play Gas-Water-Electric on the resulting graph, as witnessed below. Thus deleting any one edge from the Petersen graph results in a graph with a $K_{3,3}$ subdivision, which is therefore non-planar.



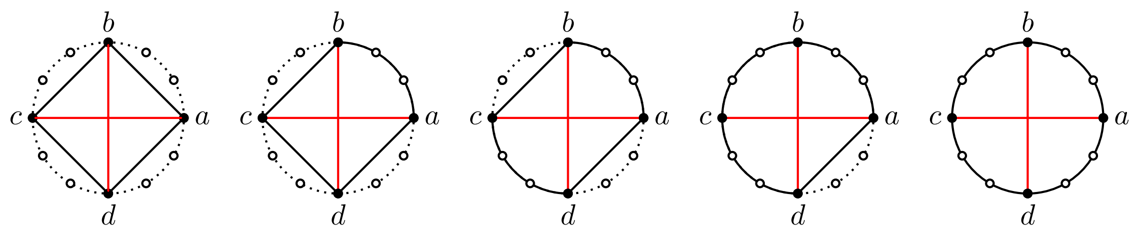
CONCLUSION: Deleting just one edge of the Petersen graph results in a graph that is still non-planar. But it is possible to delete two edges and get a planar graph.

6.2.7 Prove that a graph is outerplanar if and only if it has no subdivision of K_4 or $K_{2,3}$.

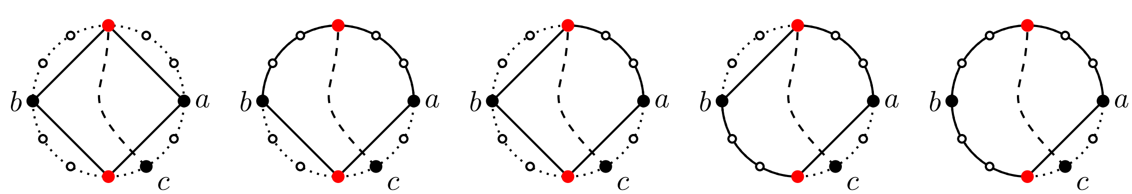
Proof: ($\Longrightarrow$) Suppose G is outerplanar. If G has any cut vertices, then any subdivision of K_5 or $K_{2,3}$ in G must occur in a block of G (because K_5 and $K_{2,3}$ have no cut vertices). Moreover any block of G must also be outer planar. (Reason: Since all of G's vertices are on the outer face of G, then all vertices of a block of G are also on the outer face of G, which is contained in the outer face of the block.)

Consequently for the rest of the proof we can assume that the outerplanar graph G is 2-connected. Give G an outer planer embedding in the plane, and denote by Z the boundary of the outer face. Thus G has no vertices inside Z, all of its edges are either on Z or inside Z, and none of them cross.

We claim that G cannot contain a K_4 subdivision. Suppose to the contrary it does have such a subdivision X. Label the vertices of degree 3 in X a, b, c, d, in that order, as we traverse Z counterclockwise. Then X must have a cycle that encounters these vertices in the same order, as indicated below (solid lines). In any event, subdivided paths from a to c and b to d must be present in X. Any internal vertices of these paths must be on the cycle Z; however, this is impossible because (for instance) the path from a to c would have to cross through either b or d, which are vertices of K_4 (and thus not vertices inserted in its subdivision). This means that the *edges* ac and bc must be present in the subdivision. However, they must cross, since they are both inside Z. This contradicts the fact that no edges of G cross.



Next we claim that G can have no subdivision of $K_{2,3}$. Suppose to the contrary that it does have such a subdivision X. Color the vertices in the partite set of size 3 in $K_{2,3}$ black, and those in the partite set of size 2 red. Color any vertex arising from the subdivision white. Now, $K_{2,3}$ has a 4-cycle with vertices alternating black and red. In X this 4-cycle appears as a cycle C with two red vertices, two black vertices, alternating red and black, with possibly some white vertices between the red and black ones. Label the black vertices a and b. Various possibilities are indicated in the drawing below, with the cycle $C \subseteq X$ drawn bold.



Now the third black vertex c of X appears somewhere on Z but not on C. In X there are paths from c to both of the red vertices. But in any case, wherever on the cycle Z the third black vertex c of X appears, the path from c to one of the red vertices (shown dashed) must cross an edge of C. This is a contradiction, so G contins no $K_{2,3}$ subdivision.

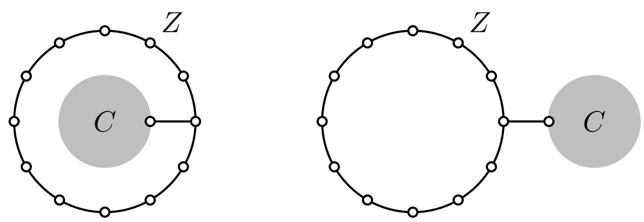
($\Longrightarrow$) For the converse we will prove the contrapositive: We will suppose G is **not** outerplanar, and show that it has a subdivision of K_4 or $K_{2,3}$.

Thus suppose that G is not outerplanar. There are two possibilities. Either G is not planar, or it is planar but cannot be drawn on the plane with all its vertices on the outer region.

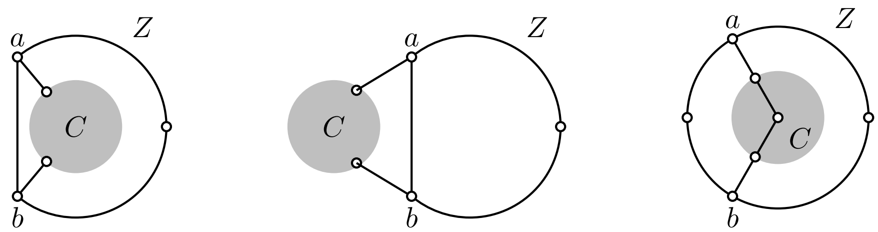
If G is not planar, then Kuratowski's theorem asserts that G has a subdivision of K_5 or $K_{3,3}$. Now, a subdivision of K_5 has a subgraph that is a subdivision of K_4, and a subdivision of $K_{3,3}$ has a subgraph that is a subdivision of $K_{2,3}$. Either way G has a subdivision of K_4 or $K_{2,3}$.

On the other hand, suppose G is planar but not outerplanar; that is, it has no planar embedding with all vertices on the outer face. Among all possible drawings of G on the plane, consider one that has the most vertices on the outer face. Then the boundary of the outer face contains a cycle Z with some of the vertices of G inside it. Let C be a component of $G - V(Z)$ that lies inside Z.

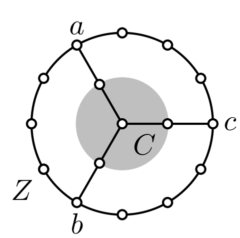
Case I: Suppose Z has only one vertex (or none at all) that is joined to C by an edge of G. (See the drawing below, left.) Then we could move C outside of Z, as shown below (right) to get a new drawing of G with even more vertices on the outside face than before. This contradicts our choice of the original drawing of G, so we conclude that Case I is impossible.



Case II: Suppose Z has exactly two vertices a, b that are joined to C by edges of G. If ab were an edge of Z (as shown below, left) then we could reflect C across the edge ab, into the outer region. (See below, center.) But this would result in a different plane drawing of G with even more vertices on the outer face than before, contrary to our choice of the original drawing of G. Therefore ab is not an edge of Z, so Z has vertices between a and b, as shown below, right. This is a $K_{2,3}$ subdivision.



Case III: Suppose Z has three (or more) distinct vertices a, b, c that are joined to C by edges of G. Then G has a K_4 subdivision as shown below.



The above cases show that G has a subdivision of $K_{2,3}$ or K_4, so the proof is complete. ∎

Here is a simpler alternate approach to Exercise 6.2.7 following West's hint.

Prove that a graph is outerplanar if and only if it has no subdivision of K_4 or $K_{2,3}$.

Proof: Suppose that G is outerplanar. Take an outerplanar embedding of G in the plane. Form a new graph G' by adding a new vertex v to the outer face of the G and connecting it to all vertices of G. The new edges can be routed from v to the vertices of G in such a way that none of them cross, so G' is planar.

Now, if G contained a subdivision of H of K_4, then then G' would contain a subdivision of K_5 consisting of the graph H with five edges joining the degree-4 vertices of H to v. This would be a subdivision of K_5 in G', contradicting the fact that G' is planar. Thus G can contain no subdivision of K_4.

Likewise, if G contained a subdivision of H of $K_{2,3}$, then then G' would contain a subdivision of $K_{3,3}$ consisting of H with three edges added from v to the three vertices of H that are the two original degree-2 vertices of the subdivision H of $K_{2,3}$. This would be a subdivision of $K_{3,3}$ in G', contradicting the fact that G' is planar. Thus G can contain no subdivision of $K_{2,3}$.

In summary, G has no subdivision of K_4 or $K_{2,3}$.

Conversely, suppose that G has no subdivision of K_4 or $K_{2,3}$. As before, form G' by adding a new vertex v that is joined to all of the vertices of G. The only difference is that, for the moment, we don't know whether G is outerplanar, or even planar. We need to prove that it is in fact outerplanar.

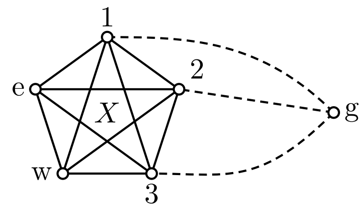
First we claim that G' is planar. Suppose to the contrary that G' is not planar. By Kuratowski's theorem it contains a subdivision of H of K_5 or $K_{3,3}$. Now, a K_5 subdivision contains a subdivision of K_4, and a $K_{3,3}$ subdivision contains a subdivision of $K_{2,3}$. Thus H cannot lie entirely in G, and therefore uses the new vertex v. Regardless of whether $v \in V(H)$ is an original vertex of K_5 or $K_{3,3}$ or one arising from the subdivision, notice that $H - v$ still contains a subdivision of K_4 or $K_{2,3}$. Since $H - v$ is a subgraph of G, we conclude that G contains a subdivision of K_4 or $K_{2,3}$. This contradicts the assumption that G contains no such subdivision, so we conclude that G' must be planar.

Now take a plane embedding of the planar graph G'. We can arrange that the vertex v is on the outer face. By construction v is adjacent to **all** vertices of G. When we remove v we get back the graph G drawn on the plane with all its vertices on the outer face. Thus G is outerplanar. ∎

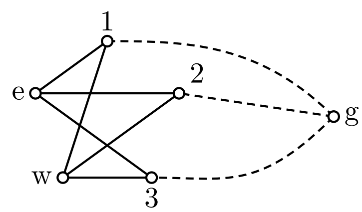
6.2.8 Prove that every 3-connected graph with at least six vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$.

Proof: Suppose that G is a 3-connected graph with at least six vertices that contains a subdivision of K_5. Call this subdivision X.

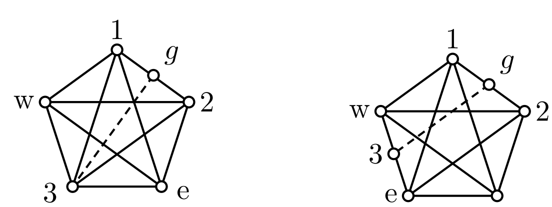
Case I: Suppose $X = K_5$, that is, X is K_5 with no edges subdivided. Since G has at least six vertices, it has a vertex g not on X. Fix three vertices of $X = K_5$ and label them $1, 2$ and 3. Label the other two vertices of X with e and w.



By the Fan Lemma (Theorem 4.2.23) there are three paths joining g to 1, 2 and 3, respectively, which intersect each other only at their starting point g. We may assume that none of these paths contains e or w, because if one of them did we could terminate it there and relabel. We now have a subdivision of $K_{3,3}$, as shown below.



Case II: Suppose that $X \neq K_5$, that is, X is K_5 with at least one subdivided edge. Label the endpoints of this edge by the numbers 1 and 2. As G is 3-connected, there must be a vertex g on the interior of this subdivided edge, and a path P from g to some other vertex of X that avoids the vertices 1 and 2. (Otherwise we could isolate g from the rest of the graph by removing the two vertices 1 and 2, contrary to the fact that G is 3-connected.) We may assume that only the endpoints of P belong to X, for if this were not the case we could terminate P at the first vertex of X that it encounters. Label the final endpoint of P with the number 3.



Notice that whether P (shown dashed above) ends at a vertex 3 that belongs to K_5 or to a subdivided edge of K_5, we can play Gas-Water-Electric on $X \cup P$. That is, we have a subdivision of $K_{3,3}$ in G. ■

6.2.11 Let H be a graph of maximum degree at most 3. Prove that a graph G contains a subdivision of H if and only if G contains a subgraph contractible to H.

($\Longrightarrow$) Suppose G contains a subdivision of H. If y is an interior vertex of a subdivided edge of H, then we can contract an edge of H that has y as an endpoint, and we are left with a subdivision of H with one less subdivided vertex as we had before. Continuing in this fashion we eventually get the graph H. Thus G contains a subgraph contractible to H. (In this direction we do not even have to use the fact that H has no vertex of degree greater than 3.)

($\Longleftarrow$) Suppose that G contains a subgraph X_0 that is contractible to H, where H has no vertex of degree greater than 3. We must show that G contains a subdivision of H as a subgraph.

Let n be the number of edge contractions that are made in transforming X_0 to H. We will use induction on n to show that G contains subdivision of H.

Basis step: If $n = 0$, then $X_0 = H$, and the subgraph X_0 of G is trivially a subdivision of H.

Inductive step: Suppose $n > 1$, and that there is a sequence of n edge contractions

$$X_0 \to X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n = H \qquad (1)$$

that results in H.

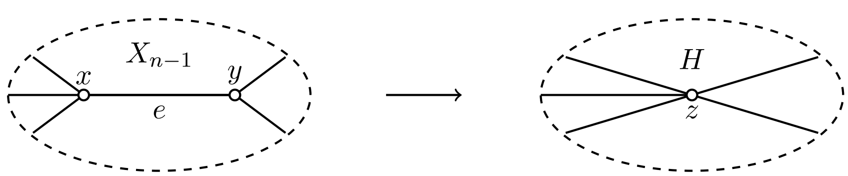
Inductive hypothesis: Suppose that whenever H' is a graph with maximum vertex degree 3, and G has a subgraph X' that is contractible to H' via fewer than n edge contractions, then G contains a subdivision of H'.

Now, X_0 is contractible to X_{n-1} via $n-1$ edge contractions, so the inductive hypothesis implies that G contains a subdivision of X_{n-1}. Call this subdivision X_{n-1}^S to distinguish it from X_{n-1}. (Note that X_{n-1}^S is a subgraph of G, though X_{n-1} may not be a subgraph of G.)

Let xy be the edge of X_{n-1} that is contracted to a vertex z in the final contraction $X_{n-1} \to H$ in (1). Notice that we have

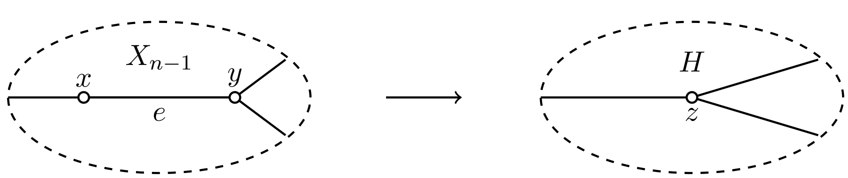
$$d_H(z) = d_{X_{i-1}}(x) + d_{X_{i-1}}(y) - 2, \qquad (2)$$

as indicated in the diagram below.



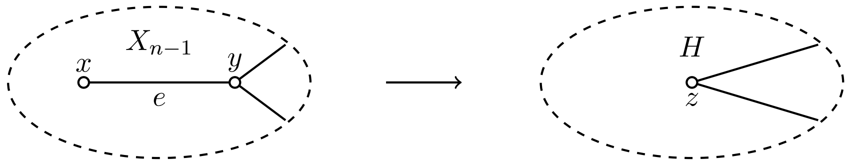
However, the vertex z of H has degree not more than 3, so Equation (2) implies that one of x or y must have degree 1 or 2 in X_{i-1} (If they both had degree 3 or greater, then (2) would yield $d_H(z) > 3$) . Without loss of generality, say that x has degree 1 or 2.

Case 1: Suppose x has degree 2, that is, $d_{X_{n-1}}(x) = 2$. In this case $1 \le d_{X_{n-1}}(y) \le 3$, as illustrated below.



In this case X_{n-1} and H are exactly the same, except that X_{n-1} has a vertex x inserted on an edge of H, so X_{n-1} is a subdivision of H. As G contains a subgraph X_{n-1}^S that is a subdivision of X_{n-1}, and X_{n-1} is a subdivision of H, it follows that G contains a subdivision of H.

Case 2: Suppose $d_{X_{n-1}}(x) = 1$, as illustrated below.



In this case $H = X_{n-1} - xy$. By the inductive hypothesis, G contains a subgraph X_{n-1}^S that is a subdivision of X_{n-1}. Just remove from X_{n-1}^S the path from x to y. What remains is a subdivision of H, hence G contains a subdivision of H. ■

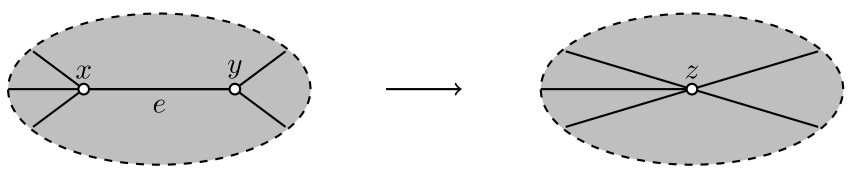
6.2.14 Prove that a graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.

(A **minor** of G is a graph obtained by applying edge contractions to a subgraph of G.)

(a) Show a minor of a planar graph is planar. Conclude that if G is planar, then it has neither K_5 nor $K_{3,3}$ as minors.

Suppose G is planar and is given a planar embedding. Then certainly any subgraph H of G now also has a planar embedding. (Just ignore the vertices and edges of G that do not appear in H, and we then have a plane representation of H.)

Further, whenever we contract an edge $e = xy$ of the plane graph H to a vertex, the result is still a plane graph. We can arrange the contraction so that the only change to H is localized to the part of the plane near the edge e, as illustrated.



Conclusion: Suppose G is planar. Then, as any minor of G is planar, G cannot have either of the non-planar graphs K_5 and $K_{3,3}$ as minors.

(b) Show that if G has neither K_5 nor $K_{3,3}$ as minors, then G is planar.

Suppose that G has neither K_5 nor $K_{3,3}$ as minors. We claim that G does not have any subgraph X that is a subdivision of K_5 or $K_{3,3}$. Indeed, if it did have such a subgraph X, then we could contract the edges in the subdivision one-by-one until X becomes a K_5 or a $K_{3,3}$. By definition, this would be a minor of G. Since G has no such minors by assumption, we infer that G has no subgraph that is a subdivision of K_5 or $K_{3,3}$. By Kuratwoski's theorem, G is planar.