Sections 10.2 Module Homomorphisms and Quotients

Definitions (Let M and N be R-modules.)

① A map $\varphi : M \to N$ is an *R-module homomorphism* if

$$\left.\begin{array}{ll}(a) & \varphi(x+y) = \varphi(x) + \varphi(y) \\ (b) & \varphi(rx) = r\varphi(x)\end{array}\right\} \text{"linear transformation properties"}$$

② Such a φ that is bijective is a *R-module isomorphism*. Under these circumstances, we write $M \cong N$.

③ $$\left.\begin{array}{l}\ker(\varphi) = \{ m \in M \mid \varphi(m) = 0 \} \leq M \\ \varphi(M) = \{ \varphi(m) \mid m \in M \} \leq N\end{array}\right\} \text{submodules (check this)}$$

④ $\mathrm{Hom}_R(M, N) = \{ \varphi \mid \varphi : M \to N \text{ is a module homomorphism} \}$

Proposition 2

① $\varphi : M \to N$ is an R-module homomorphism $\iff \varphi(rx+y) = r\varphi(x) + \varphi(y)$ for all $r \in R$, $x, y \in M$.

② $\mathrm{Hom}_R(M, N)$ is an abelian group, where $\varphi + \theta : M \to N$ is map $(\varphi + \theta)(m) = \varphi(m) + \theta(m)$.

If R is commutative, then $\mathrm{Hom}_R(M, N)$ is an R-module, when $r\varphi : M \to N$ is $(r\varphi)(m) = r\varphi(m)$

Note: commutativity necessary:

$$\begin{aligned}(r\varphi)(sm) &= r\varphi(sm) \\ &= rs\varphi(m) \\ &= sr\varphi(m) \\ &= s(r\varphi)(m)\end{aligned}$$

needs to come out front

③ $\varphi \in \mathrm{Hom}_R(M, N)$ and $\theta \in \mathrm{Hom}_R(N, L) \implies \theta \circ \varphi \in \mathrm{Hom}(M, L)$

④ $\mathrm{Hom}_R(M, M)$ is a ring, where mult. is composition.

Note: Distributive property $\varphi\circ(\mu+\lambda) = \varphi\circ\mu + \varphi\circ\lambda$ comes from the fact $\varphi(x+y) = \varphi(x)+\varphi(y)$.

Note If R is commutative, $\mathrm{Hom}_R(M,M)$ is also an R-algebra. Homomorphism $f: R \to \mathrm{Hom}_R(M,M)$ is defined as $f(r) = r\,\mathrm{id}$. This is in the centre of $\mathrm{Hom}_R(M,M)$ because $\varphi(r)\circ\varphi = (r\,\mathrm{id})\circ\varphi = \varphi\circ(r\,\mathrm{id})$ [check].

Definitions An *endomorphism* is a module homomorphism $M \to M$ (i.e. domain = co-domain)

$\mathrm{End}(M) = \mathrm{Hom}_R(M,M)$ is called the *endomorphism ring of M*.

Quotients of Modules

Proposition 3 Suppose M is an R-module and $N \subseteq M$ is a submodule. Then the abelian group M/N is an R-module with R action

$$r(x+N) = rx + N.$$

Moreover, the projection $\pi: M \to M/N$, where $\pi(x) = x+N$ is an R-module homomorphism with kernel N.

Theorem 4 (Isomorphism Theorems)

① If $\varphi : M \to N$ is an R-module homomorphism, Then $M/\ker\varphi \cong \varphi(M)$.

Specifically,

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[dddrr, mapsto, "\pi"'] & & & \varphi(x) \\
& M \arrow[rr, "\varphi"] \arrow[dr] & & \varphi(M) \\
& & M/\ker\varphi \arrow[ur, "\sim"'] & \\
& & x+\ker\varphi \arrow[uuur, mapsto] &
\end{tikzcd}$$

② } Read in text.
③

④ Suppose N is an R-submodule of M

$$\left\{ \begin{array}{c} \text{submodules } A \\ N \subseteq A \subseteq M \end{array} \right\} \xleftrightarrow{\text{Bijection}} \left\{ \begin{array}{c} \text{submodules} \\ \text{of } M/N \end{array} \right\}$$

$$A \longleftrightarrow A/N \subseteq M/N.$$