

Sections 10.2 Module Homomorphisms and Quotients

Definitions (Let M and N be R -modules.)

① A map $\varphi: M \rightarrow N$ is an R -module homomorphism if

$$(a) \quad \varphi(x+y) = \varphi(x) + \varphi(y)$$

$$(b) \quad \varphi(rx) = r\varphi(x)$$

} "linear transformation properties"

② Such a φ that is bijective is a R -module isomorphism. Under these circumstances, we write $M \cong N$.

③ $\ker(\varphi) = \{m \in M \mid \varphi(m) = 0\} \subseteq M$
 $\varphi(M) = \{\varphi(m) \mid m \in M\} \subseteq N$ } submodules (check this)

④ $\text{Hom}_R(M, N) = \{ \varphi \mid \varphi: M \rightarrow N \text{ is a module homomorphism} \}$

Proposition 2

① $\varphi: M \rightarrow N$ is an R -module homomorphism $\Leftrightarrow \varphi(rx+y) = r\varphi(x) + \varphi(y)$
for all $r \in R, x, y \in M$.

② $\text{Hom}_R(M, N)$ is an abelian group, where $\varphi + \theta: M \rightarrow N$
is map $(\varphi + \theta)(m) = \varphi(m) + \theta(m)$.

If R is commutative, then $\text{Hom}_R(M, N)$ is an R -module,
where $r\varphi: M \rightarrow N$ is $(r\varphi)(m) = r\varphi(m)$

Note: commutativity necessary:

$$\begin{aligned} (r\varphi)(sm) &= r\varphi(sm) \\ &= rs\varphi(m) \\ &= sr\varphi(m) \\ &= s(r\varphi)(m) \end{aligned}$$

needs to come out front

③ $\varphi \in \text{Hom}_R(M, N)$ and $\theta \in \text{Hom}_R(N, L) \Rightarrow \theta \circ \varphi \in \text{Hom}(M, L)$

④ $\text{Hom}_R(M, M)$ is a ring, where mult. is composition.

Note: Distributive property $\varphi \circ (\mu + \lambda) = \varphi \circ \mu + \varphi \circ \lambda$
comes from the fact $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Note If R is commutative, $\text{Hom}_R(M, M)$ is also an R -algebra.

Homomorphism $\varphi: R \rightarrow \text{Hom}_R(M, M)$ is defined as
 $\varphi(r) = r \text{ id}$. This is in the center of $\text{Hom}_R(M, M)$
because $\varphi(r) \circ \varphi = (r \text{ id}) \circ \varphi = \varphi \circ (r \text{ id})$ [check].

Definitions An endomorphism is a module homomorphism
 $M \rightarrow M$ (i.e. domain = co-domain)

$\text{End}(M) = \text{Hom}_R(M, M)$ is called the endomorphism ring of M .

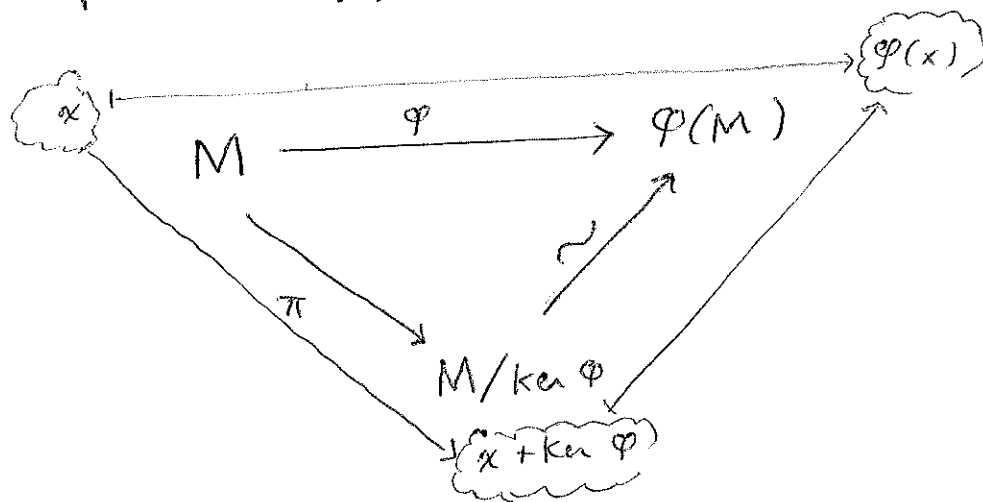
Quotients of Modules

Proposition 3 Suppose M is an R -module and
 $N \subseteq M$ is a submodule. Then the abelian
group M/N is an R -module with R action
 $r(x+N) = rx + N$.

Moreover, the projection $\pi: M \rightarrow M/N$, where
 $\pi(x) = x+N$ is an R -module homomorphism
with kernel N .

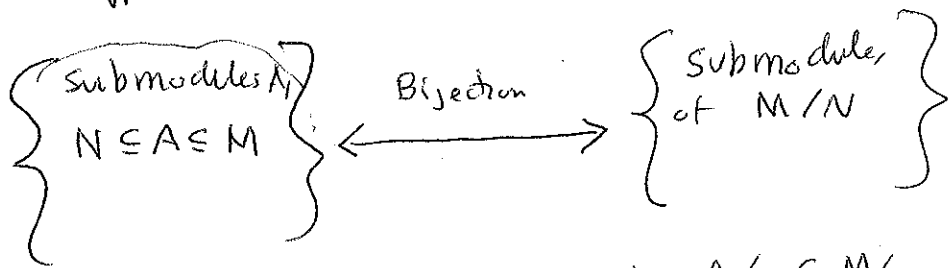
Theorem 4 (Isomorphism Theorems)

- ① If $\varphi: M \rightarrow N$ is an R -module homomorphism, then $M/\ker \varphi \cong \varphi(M)$.
Specifically,



- ② } Read in text.
③ }

④ Suppose N is an R -submodule of M



$$A \longleftrightarrow A/N \subseteq M/N.$$

