Part III Modules and Vector Spaces

Chapter 10 Introduction to Module Theory.

Section 10.1 Basic Definitions

Definition Suppose R is a ring. A (left) R-module is a set M with

1. A binary operation $+$ on M for which M is an abelian group,
2. An action of R on M, denoted rm, such that

   (a) $(r+s)m = rm + sm$

   (b) $(rs)m = r(sm)$

   (c) $r(m+n) = rm + rn$

   (d) $1m = m$ .... If R has 1.

Example $R = \mathbb{R}$. An $\mathbb{R}$-module is a vector space over $\mathbb{R}$
A vector space over $\mathbb{R}$ is an $\mathbb{R}$ module

$$\left\{ \begin{array}{l} \mathbb{F}\text{-modules} \\ (\mathbb{F} \text{ a field}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{vector spaces} \\ \text{over } \mathbb{F} \end{array} \right\} \qquad (\mathbb{F} \text{ is a field})$$

Example $R = \mathbb{Z}$. A $\mathbb{Z}$-module is an abelian group satisfying (a) – (d). But these are standard exponential properties of any abelian group.

$$\left\{ \mathbb{Z}\text{-modules} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{abelian} \\ \text{groups} \end{array} \right\}$$

Example R is a ring. $R^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in R\}$
with action $r(a_1, a_2, \dots, a_n) = (ra_1, ra_2, \dots, ra_n)$
Then R^n is an R-module.
This is a generalization of a vector space.
(finite dimensional)

<u>Example</u> $F[x]$ - modules, where F is a field.
(This example leads to 90% of advanced linear algebra.)

Suppose: $\boxed{M \text{ is a vector space over } F, \quad T: M \to M \text{ is a linear transformation}}$

Then $T^k = \underbrace{T \circ T \circ T \circ \cdots \circ T}_{k}$ is well-defined lin. trans $T^k : M \to M$.

Given $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$,
let $f(T) = a_0 I + a_1 T + \cdots + a_n T^n$ be the lin. trans $f(T): M \to M$
where $f(T)(m) = a_0 I m + a_1 T m + a_2 T^2 m + \cdots + a_n T^n m$

Then $F[x]$ acts on M as $f(x)m = f(T)(m)$

Check (a) – (d) hold

Thus $\boxed{M \text{ is an } F[x] \text{ module}}$

$$\left\{ \begin{array}{l} \text{vector spaces } M \text{ and} \\ \text{lin. trans } T: M \to M \end{array} \right\} \rightsquigarrow \left\{ F[x] \text{ modules} \right\}$$

Conversely, let M be an arbitrary $F[x]$ module.
Then $F \subseteq F[x]$ acts on M, and (a) – (d) hold.
This makes M a vector space over F.

Moreover, action of $x \in F[x]$ on M is a linear trans.

$$\begin{aligned} x(rm + sn) &= x(rm) + x(sn) \\ &= (xr)m + (xs)n \\ &= (rx)m + (sx)n \\ &= r(xm) + s(xn) \end{aligned}$$

$$T(rm + sn) = rTm + sTn$$

$$\left\{ \begin{array}{l} \text{vector spaces } M \text{ and} \\ \text{lin. trans } T: M \to M \end{array} \right\} \leftsquigarrow \left\{ F[x] \text{ modules} \right\}$$

Therefore $\left\{ \begin{array}{l} \text{vector spaces } M \text{ and} \\ \text{lin. trans. } T: M \to M \end{array} \right\} \longleftrightarrow \left\{ F[x] \text{ modules} \right\}$

Thus module theory will help explain

(a) structure of abelian groups.

(b) structure of linear transformations

(c) many other things.

Definition:

An R-submodule of an R-module M is a subgroup $N \leq M$ that is closed under the action of R, i.e. $r \in R$, $n \in N \Rightarrow rn \in N$.

Proposition 1 (Submodule Criterion)

Let M be an R-module and $N \subseteq N$. Then N is a R-submodule $\Leftrightarrow$

(1) $N \neq \phi$.

(2) $r \in R$, $x, y \in N \Rightarrow x + ry \in N$.

ALGEBRAS

Def. Suppose R is a commutative ring with 1.
An R-algebra is a ring A with 1, together with a homomorphism $f : R \longrightarrow A$, where $f(1_R) = 1_A$ and $f(R) \subseteq Z(A)$.

Note: An R-algebra is an R-module with $ra = f(r)a = a f(r) = a \cdot r$

Also has non-module property $r(ab) = (ra)b = a(rb)$.

Example $R[x_1, x_2, \cdots, x_n]$ when $f : R \to R[x_1 x_2 \cdots x_k]$, $f(r) = r$ (const. poly)

Example $M_2(\mathbb{R})$ when $f : \mathbb{R} \to M_2(\mathbb{R})$, $f(r) = rI$.

$$r.X = f(r)X = rIx = \underbrace{rX}_{\text{scalar mult.}}$$