

Part III Modules and Vector Spaces

Chapter 10 Introduction to Module Theory.

Section 10.1 Basic Definitions

Definition Suppose R is a ring. A (left) R -module is a set M with

① A binary operation $+$ on M for which M is an abelian group,

② An action of R on M , denoted rm , such that

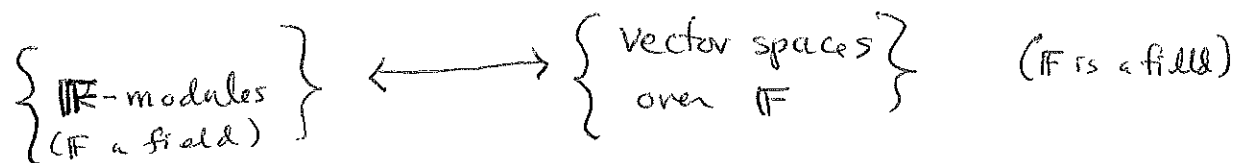
(a) $(r+s)m = rm + sm$

(b) $(rs)m = r(sm)$

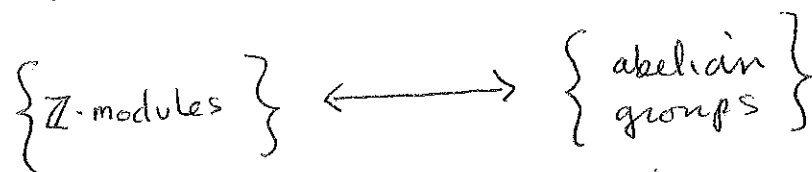
(c) $r(m+n) = rm + rn$

(d) $1m = m$ ----- If R has 1 .

Example $R = \mathbb{R}$. An \mathbb{R} -module is a vector space over \mathbb{R}
A vector space over \mathbb{R} is an \mathbb{R} module



Example $R = \mathbb{Z}$. A \mathbb{Z} -module is an abelian group satisfying (a) - (d). But these are standard exponential properties of any abelian group.



Example R is a ring. $R^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in R\}$
with action $r(a_1, a_2, \dots, a_n) = (ra_1, ra_2, \dots, ra_n)$
Then R^n is an R -module.

This is a generalization of a vector space.
(finite dimensional)

Example $F[x]$ -modules, where F is a field.

(This example leads to 90% of advanced linear algebra.)

Suppose: M is a vector space over F , $T: M \rightarrow M$ is a linear transformation

Then $T^k = \underbrace{T \circ T \circ T \circ \dots \circ T}_k$ is well-defined lin. trans $T^k: M \rightarrow M$.

Given $f(x) = a_0 + a_1x + \dots + a_nx^n \in F[x]$,

let $f(T) = a_0I + a_1T + \dots + a_nT^n$ be the lin. trans $f(T): M \rightarrow M$

where $f(T)(m) = a_0I_m + a_1Tm + a_2T^2m + \dots + a_nT^nm$

Then $F[x]$ acts on M as $f(x)m = f(T)(m)$

Check (a)-(d) hold

Thus M is an $F[x]$ module

$\left. \begin{array}{l} \text{vector spaces } M \text{ and} \\ \text{lin. trans } T: M \rightarrow M \end{array} \right\} \rightsquigarrow \left\{ F[x] \text{ modules} \right\}$

Conversely, let M be an arbitrary $F[x]$ module.

Then $F \subseteq F[x]$ acts on M , and (a)-(d) hold.

This makes M a vector space over F .

Moreover, action of $x \in F[x]$ on M is a linear trans.

$$\begin{aligned} x(rm + sn) &= x(rm) + x(sn) \\ &= (xr)m + (xs)n \\ &= (rx)m + (sx)n \\ &= r(xm) + s(xn) \\ \hline T(rm + sn) &= rTm + sTn \end{aligned}$$

$\left. \begin{array}{l} \text{vector spaces } M \text{ and} \\ \text{lin. trans } T: M \rightarrow M \end{array} \right\} \longleftarrow \left\{ F[x] \text{ modules} \right\}$

Therefore $\left. \begin{array}{l} \text{vector spaces } M \text{ and} \\ \text{lin. trans. } T: M \rightarrow M \end{array} \right\} \longleftrightarrow \left\{ F[x] \text{ modules} \right\}$

Thus module theory will help explain

- (a) structure of abelian groups.
- (b) structure of linear transformations
- (c) many other things.

Definition:

An R -submodule of an R -module M is a subgroup $N \subseteq M$ that is closed under the action of R , i.e. $r \in R, n \in N \Rightarrow rn \in N$.

Proposition 1 (Submodule Criterion)

Let M be an R -module and $N \subseteq M$. Then N is a R -submodule \Leftrightarrow

(1) $N \neq \emptyset$.

(2) $r \in R, x, y \in N \Rightarrow x + ry \in N$.

ALGEBRAS

Def. Suppose R is a commutative ring with 1. An R -algebra is a ring A with 1, together with a homomorphism $f: R \rightarrow A$, where $f(1_R) = 1_A$ and $f(R) \subseteq Z(A)$.

Note: An R -algebra is an R -module with $ra = f(r)a = a f(r) = a \cdot r$. Also has non-module property $r(ab) = (ra)b = a(rb)$.

Example $R[x_1, x_2, \dots, x_n]$ where $f: R \rightarrow R[x_1, x_2, \dots, x_n]$, $f(r) = r$ (const. poly)

Example $M_2(\mathbb{R})$ where $f: \mathbb{R} \rightarrow M_2(\mathbb{R})$, $f(r) = rI$.

$$r \cdot X = f(r)X = rIX = \underbrace{rX}_{\text{scalar mult.}}$$