Section 9.3 Polynomial Rings that are UFDs

Recall:

- $R[x_1, x_2, \ldots, x_n] = R[x_1, x_2, \ldots, x_{n-1}][x_n]$
- Any I.D. R has a field of fractions $F = \{\frac{a}{b} \mid a, b \in R, b \neq 0\}$, $R \subseteq F$.
- Corollary 2 Given $I \subseteq R$, $R[x]/(I) \cong R/I\,[x]$
  If I prime in R, then (I) prime in $R[x]$

We've seen how properties of R influence properties of $R[x]$.

- R is a field $\Rightarrow$ $R[x]$ is ED, PID, UFD.
- R is an ID $\Leftrightarrow$ $R[x]$ is an ID.
  $\Leftrightarrow$ $R[x_1, x_2, \ldots, x_n]$ is an ID

Today's Goal:

Theorem 7 $\quad R$ is a UFD $\Longleftrightarrow$ $R[x]$ is a UFD

Corollary 8 $\quad R$ is a UFD $\Longleftrightarrow$ $R[x_1, x_2, x_3, \ldots, x_n]$ is a UFD.

The following question is a key to establishing these results. It is answered affirmatively by the so-called Gauss Lemma.

Question: If $f(x) \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$ factors in $\mathbb{Q}[x]$, does it factor in $\mathbb{Z}[x]$?
If $f(x) \in R[x] \subseteq F[x]$ factors in $F[x]$, does it factor in $R[x]$?

Proposition 5 Gauss' Lemma

Let R be an ID with field of fractions F $\{R = \mathbb{Z},\ F = \mathbb{Q}\}$

If $p(x) \in R[x]$ is reducible in $F[x]$, then it's reducible in $R[x]$.

Specifically if $\underbrace{p(x)}_{R[x]} = \underbrace{A(x)\,B(x)}_{F[x]}$

then $\exists\ r, s \in F$ such that

$$\underbrace{p(x)}_{R[x]} = \underbrace{rA(x) \cdot sB(x)}_{R[x]}.$$

[Note: necessarily $rs = 1$]

Proof (outline)

Suppose $\underbrace{p(x)}_{R[x]} = \underbrace{A(x)\ B(x)}_{F[x]}$.

Then $\underbrace{de\, p(x)}_{R[x]} = \underbrace{d\,A(x)\ e\,B(x)}_{R[x]}$ for some $d, e \in R$

So $\underbrace{p_1 p_2 p_3 \cdots p_k}_{\text{prime factoring of } de} p(x) = d\,A(x)\ e\,B(x)$ $(*)$

Thus $(p_1) \subseteq R\cancel{[x]}$ is prime ideal in R.

Corollary 2: $R[x]/((p_1)) = R[x]/p_1R[x] \cong R/(p_1)[x]$

Then $R/(p_1)$ is ID, $\Rightarrow$ $R[x]/p_1R[x]$ is I.D.

Note: $d\,A(x) \cdot e\,B(x) + p_1 R[x] = 0 + p_1 R[x]$ in $R[x]/p_1R[x]$

$\overline{d\,A(x)}\ \overline{e\,B(x)} = \overline{0}$

Say $\overline{d\,A(x)} = \overline{0}$. $\rightsquigarrow$ $d\,A(x) \in p_1 R[x] \rightsquigarrow \frac{1}{p_1} d\,A(x) \in R[x]$

$(*) \rightsquigarrow p_2 p_3 \cdots p_k\ p(x) = \underbrace{\frac{1}{p_1} d\,A(x)}_{R[x]}\ \underbrace{eB(x)}_{R[x]}$

Continue process with p_2 instead of p_1, etc.

Get: $p(x) = \underbrace{r\,A(x) \cdot s\,B(x)}_{R[x]}$. $\blacksquare$

<u>Theorem 7</u> R is UFD $\iff$ R[x] is UFD

<u>Proof</u> ($\Leftarrow$) Trivial because $R \subseteq R[x]$.

($\Rightarrow$) Basic Idea: $R[x] \subseteq F[x]$

Suppose $p(x) \in R[x]$. Need to show $p(x)$ factors uniquely.
Note That $p(x) \in F[x]$ (UFD)
Unique factorization in $F[x]$:

$$p(x) = q_1(x)\, q_2(x) \cdots q_k(x).$$

Now use Gauss' Lemma to convert This to a unique factorization in $R[x]$.

<u>Corollary 8</u> R is UFD $\iff$ R[x] is UFD.

<u>Proof</u> Follows from Theorem 7 and
$R[x_1, x_2, \ldots, x_n] = R[x_1, x_2, \ldots, x_{n-1}][x_n]$