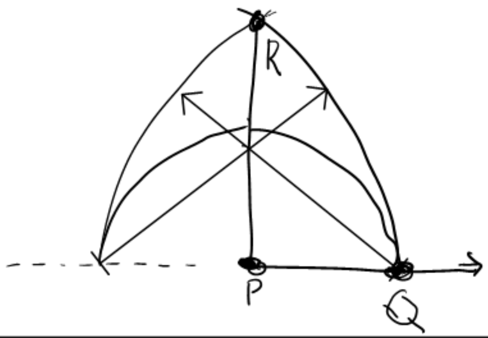
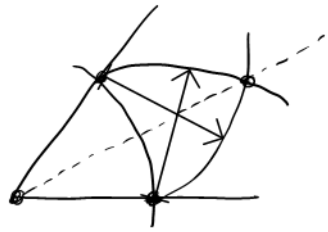
Section 13.3 Classical Ruler-and-Compass Constructions

Classical Greek geometry was concerned with constructions done only with an (unmarked) ruler and compass (physical manifestations of lines and circles.

Example Erect a perpindicular to $\overrightarrow{PQ}$

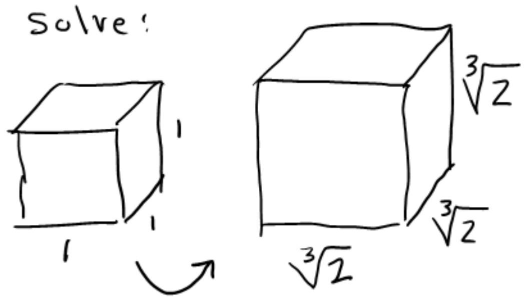


Example Bisect an angle.



The greeks developed an extensive body of such constructions. But there were three problems they could not solve:

I. Doubling the Cube: Given the side length of a cube, construct the side lenght of a cube with twice the volume.



II. Trisect a given angle

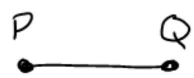
III Squaring the circle: Given a circle, construct a square with the same area

These turned out to be impossible, but this was not understood until the development of field theory in the 1880's

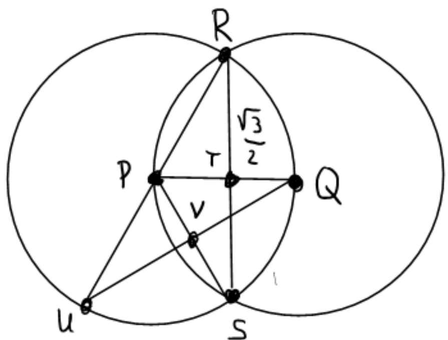
Todays Goal Understand why they are impossible. (i.e. impossible using only ruler and compass.)

Construction Procedure

Begin with a segment PQ of unit length:



Other points are determined by intersections of circles of radius PQ centered at P or Q or by intersections of lines through points thus obtained, or by intersections of circles and lines thus obtained, etc.



Definitions

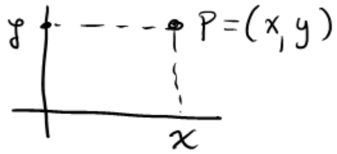
Constructible Point: Point obtained this way

Constructible Segment: Segment joining constrvcible points

Constructible Number: Length of a constructible segment

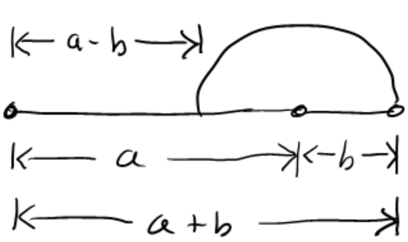
Ex. $1, \ 1/2, \ 2, \ \sqrt{3}/2$, etc.

Proposition $P=(x,y)$ is a constructible point $\iff$ x, y are constructible numbers



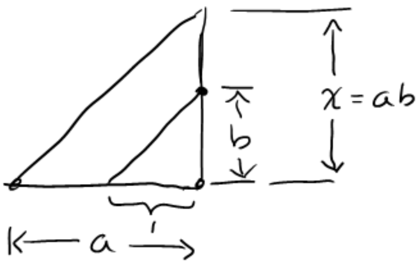
Proposition Constructible numbers (and their negatives) constitute a subfield $\mathcal{C} \subseteq \mathbb{R}$

Proof



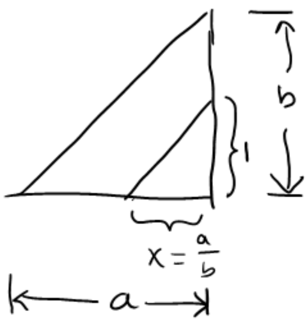
sum/difference of constructible numbers is constructible.

$$\frac{b}{1} = \frac{x}{a} \qquad x = ab$$



product of constructible numbers is constructible.

$$\frac{x}{1} = \frac{a}{b}$$

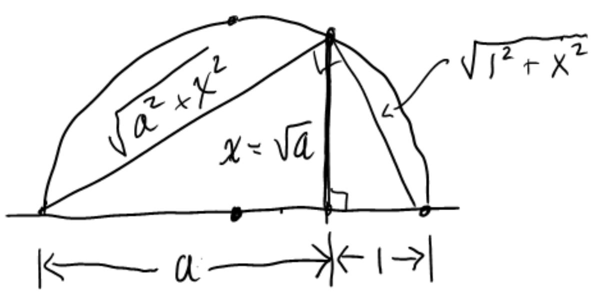


Quotient of constructible numbers is constructible

Proposition The square root of a constructible number is constructible

Proof Suppose a is constructible

Get $\sqrt{a}$ with ruler & compass:



Pythagorean Theorem:

$$\sqrt{a^2+x^2}^{\,2} + \sqrt{1^2+x^2}^{\,2} = (a+1)^2$$

$$a^2+x^2+1+x^2 = a^2+2a+1$$

$$2x^2 = 2a$$

$$x^2 = a$$

$$x = \sqrt{a}$$

Corollary The field $\mathcal{C} \subseteq \mathbb{R}$ of constructible numbers has subfields:

$$F_0 = \mathbb{Q} \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq \mathcal{C}$$

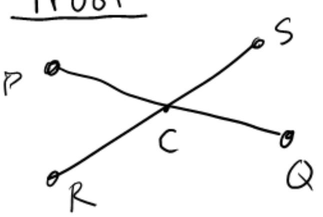
$F_1 = \langle \mathbb{Q} \cup \{\sqrt{x} \mid x \in \mathbb{Q}\} \rangle$, $F_2 = \langle F_1 \cup \{\sqrt{x} \mid x \in F_1\} \rangle$ etc.

$$\boxed{F_{k+1} = \langle F_k \cup \{\sqrt{x} \mid x \in F_k\} \rangle}$$

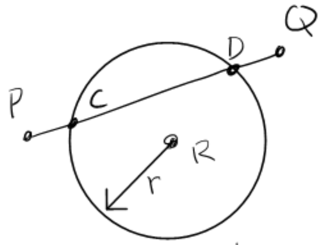
e.g. $\frac{1+\sqrt{5}}{2}$ e.g. $\frac{5}{2} + \sqrt{2+\sqrt{7}}$

Proposition $\mathcal{C} = \bigcup_{i=0}^{\infty} F_i$

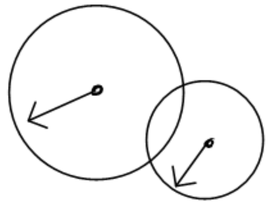
Proof



Coordinates of P, Q, R, S in $F_k \Rightarrow$ coordinates of C in F_k



r & coordinates of P, Q, R in $F_k \Rightarrow$ coordinates of C, D are in F_{k+1}



If radii and coordinates of centers are in F_k then intersections are in F_{k+1}

Series of such operations beginning with points with coordinates in $F_0 = \mathbb{Q}$ yields points with coordinates in some F_n

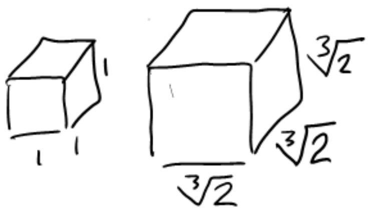
Therefore any constructible number is in some subfield F_n ∎

Proposition 23 If α is obtained from points in F_k by a series of ruler-and-compass constructions, then

$$[F_k(\alpha) : F_k] = 2^m \text{ for some } m.$$

Thus the degree over $\mathbb{Q}$ of any constructible # is a power of 2

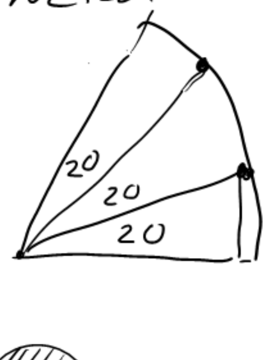
Theorem Doubling the cube is impossible.



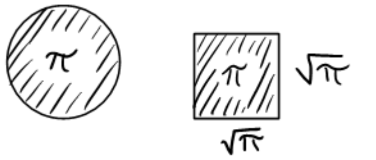
Proof This would involve constructing $\alpha = \sqrt[3]{2}$. But $m_\alpha(x) = x^3 - 2$ has degree 3 over $\mathbb{Q}$, so $\sqrt[3]{2}$ can't be constructed.

Theorem Trisecting an angle is impossible

Proof If this could be done, we could trisect a 60° angle. Then $\alpha = 2\cos 20$ would be constructible. Text shows $m_\alpha(x) = x^3 - 3x - 1$. Thus α has degree 3, hence not constructible.



Theorem Squaring the circle is impossible



Proof: Take circle of radius 1, area π. We need to construct $\alpha = \sqrt{\pi}$. Note $[\mathbb{Q}(\sqrt{\pi}) : \mathbb{Q}] = \infty$ is not a power of 2. Thus can't construct $\sqrt{\pi}$.