Chapter 11 Vector Spaces

11.3 6 7
11.2 9, 11

11.1 Basic Theory

A vector space V over a field F is an F-module V.

A subspace W of a v.s. V is a submodule W of V. $\varphi(V)$ is subspace of U
A linear trans $\varphi: V \to U$ is an F-module Homo. Ker φ is subspace of V,
The Span of a set $A \subseteq V$ is $\text{span}(A) = FA = \{\sum_{\text{finite}} x_i a_i \mid x_i \in F, a_i \in A\}$

If span(A) = V, A is called a spanning set.
A linearly independent spanning set is called a basis for V.

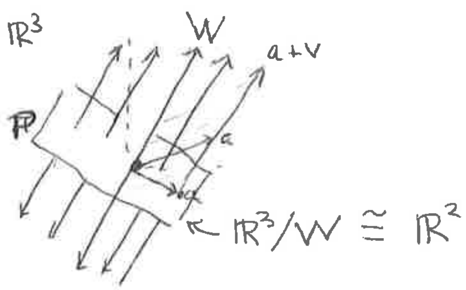
Corollary 4 (Corollary to Theorem 3 [Replacement Theorem])

1. Suppose V has a finite basis $\{a_1\ a_2 \ldots a_n\}$
   Any linearly independent set has at most n elements
   Any spanning set has at least n elements
2. If V has a finite basis, then any two bases have the same cardinality.

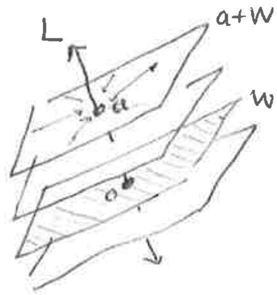
Definition If V has a finite basis, the number of elements in any basis is called the dimension of V; denoted $\dim_F(V)$,

Theorem If $\dim(V) = n$, then $V \cong F^n$. Thus any two vector spaces of the same dimension are isomorphic

Quotients: If W is a subspace of V, then V/W is a v.s. over F



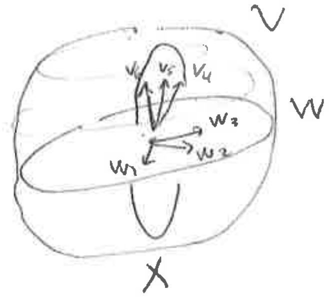
Cosets a+W are parallel lines



Cosets a+W are parallel planes

Let $\varphi: \mathbb{R}^3 \to P$ be orthogonal projection to P
Ker φ = W
so $\mathbb{R}^3/W \cong P \cong \mathbb{R}^2$

Let $\varphi: \mathbb{R}^3 \to L$ be orthogonal projection to L
Ker φ = W
$\mathbb{R}^3/W \cong L \cong \mathbb{R}$



<u>Theorem 7</u> Suppose W is subspace of V

Then $\dim W = \dim W + \dim V/W$

<u>Proof</u>: Basis for V: $A = \{ \underbrace{\overbrace{w_1\, w_2\, w_3 \cdots w_m}^{n}}_{\text{basis for } W},\ \overbrace{v_{m+1}\, v_{m+2} \cdots v_n}^{m-n} \}$

Then $\sum x_i v_i \notin W$ (unless all $x_i = 0$)

So $B = \{ v_m + W,\ v_{m+1} + W, \ldots\ v_n + W \}$ is lin. ind in V/W.

Also spans V/W because any $x + W = \sum x_i w_i + \sum x_j v_j + W = \sum x_i v_i + W$

Then $\dim W = m$, $\dim V/W = n - m = \dim V - \dim W$

$\Rightarrow \dim V = \dim W + \dim V/W$.

<u>Corollary 8</u> Given lin trans $\varphi : V \to U$,

$\dim V = \dim \ker \varphi + \dim \varphi(V)$

<u>Proof</u> $V/\ker\varphi \cong \varphi(V)$

$\dim (V/\ker\varphi) = \dim \varphi(V)$

$\dim V - \dim \ker\varphi = \dim \varphi(V)$

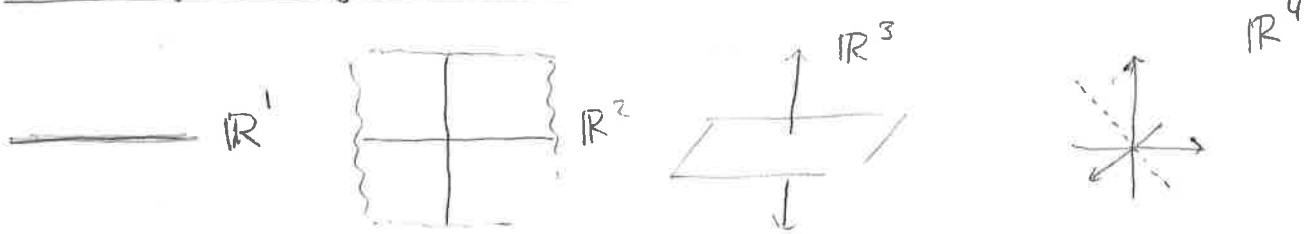
$\dim V = \dim \ker\varphi + \dim \varphi(V)$.

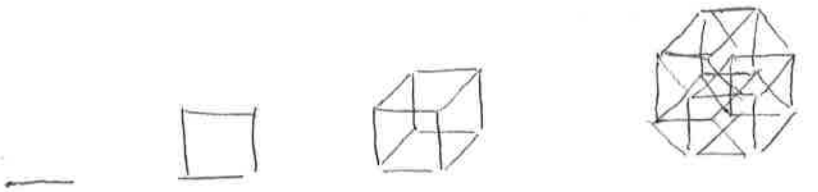
<u>Corollary 9</u> Suppose $\varphi : V \to W$ is a lin trans, where $\dim V = \dim W$

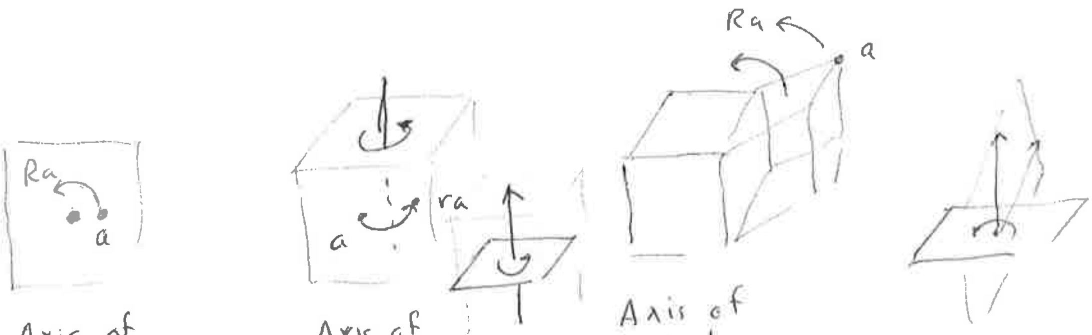
The following are equivalent

1. φ is an isomorphism
2. φ is injective
3. φ is surjective
4. φ sends basis of V to basis of W.

Visualizing higher dimensional space







Axis of rotation in $\mathbb{R}^2$ is point

Axis of rotation in $\mathbb{R}^3$ is a line

Axis of rotation in $\mathbb{R}^4$ is a plane.

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$