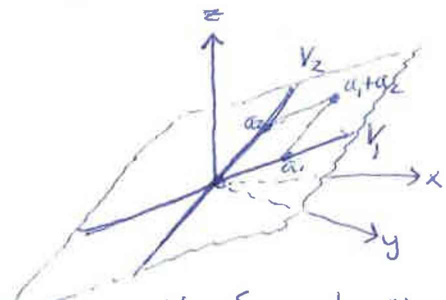
Section 10.3 Generation of Modules, Direct Sums, Free Modules.

In what follows we compare structures in an R-module M to vector spaces over $\mathbb{R}$. All the "new" terms introduced here are entirely parallel to old ideas involving vector spaces.

VECTOR SPACES OVER $\mathbb{R}$	R-Modules M.

Sum of subspaces



$V_1+V_2=\{a_1+a_2 \mid a_1\in V_1,\ a_2\in V_2\}$

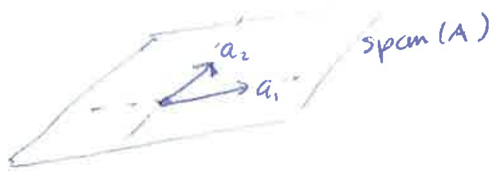
Sum of sub-modules

If $N_1, N_2, \dots, N_k$ are submodules of M, then their sum is

$N_1+N_2+\dots+N_i=\{a_1+a_2+\dots+a_i \mid a_i\in N_i\ \forall i\}$

Span of a set $A=\{a_1, a_2 \dots a_k\}$

$\text{Span}(A)=\{\sum r_i a_i \mid r_i\in\mathbb{R},\ a_i\in A\}$



Submodule generated by $A\subseteq M$.

If $A\subseteq M$, the submodule generated by A is

$RA=\{r_1a_1+r_2a_2+\dots+r_ma_m \mid r_i\in R,\ a_i\in A\}$

If $N=RA$, we say N is generated by A. A is a set of generators for N.

Finite dimensional space

If A is finite, then span(A) is a finite dimensional vector space.

$N\subseteq M$ is finitely generated if $N=RA$ for some finite set A. Similarly, M is finitely generated if $M=RA$ for some finite $A\subseteq M$.

One dimensional space

A subspace V is 1-D if $V=\text{span}\{a\}$ for some vector a

$N\subseteq M$ is cyclic if $\exists\ a\in M$ such that $N=Ra=\{ra \mid r\in R\}$

Direct Sums and Direct Products

Definition If $M_1, M_2 \dots M_k$ are R-modules, their direct product is

$$M_1 \times M_2 \times \dots \times M_k = \{(m_1, m_2, \dots, m_k) \mid m_i \in M_i \ \forall i\}.$$

This is an abelian group and an R-module under action

$$r(m_1, m_2, \dots, m_k) = (rm_1, rm_2, \dots, rm_k).$$

Alternate Notation:

$$\underbrace{M_1 \times M_2 \times \dots \times M_k}_{\text{direct product}} = \underbrace{M_1 \oplus M_2 \oplus \dots \oplus M_k}_{\text{direct sum}}.$$

These are the same if k is finite. The difference emerges when there are an ∞ number of factors.

Direct Product

$$M_1 \times M_2 \times M_3 \times \dots = \{(m_1, m_2, m_3 \dots) \mid m_i \in M_i \ \forall i\}$$

Direct Sum

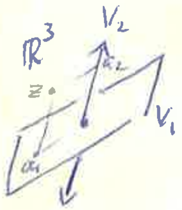
$$M_1 \oplus M_2 \oplus M_3 \oplus \dots = \{(m_1, m_2, m_3 \dots) \mid m_i \in M_i \ \forall i \text{ and } \text{~~all but~~} \ m_i = 0 \text{ for all but finitely many } i\}$$

Thus $\bigoplus_{i=1}^{\infty} M_i \subseteq \prod_{i=1}^{\infty} M_i$

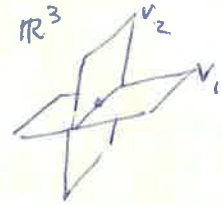
$$\underbrace{\bigoplus_{i \in I} M_i}_{\text{Direct sum}} \subseteq \underbrace{\prod_{i \in I} M_i}_{\text{Direct Product}}$$

A decomposition theorem

For the next result, keep the following vector space picture in mind. The proof mirrors the (almost obvious) vector space setting.



- $\mathbb{R}^3 \cong V_1 \times V_2 = V_1 + V_2 = \mathbb{R}^3$
- $V_1 \cap V_2 = \{0\}$
- Any $z \in \mathbb{R}^3$ has a unique expression $z = a_1 + a_2$, $a_1 \in V_1$, $a_2 \in V_2$



- $\mathbb{R}^3 \not\cong V_1 \times V_2 \not\cong V_1 + V_2 = \mathbb{R}^3$
- $V_1 \cap V_2 \neq \{0\}$
- Many ways to write $z = a_1 + a_2$

Proposition 5 Suppose $N_1\ N_2 \dots N_k$ are submodules of the R-module M. Then the following are equivalent.

1. $\pi : N_1 \times N_2 \times \dots \times N_k \longrightarrow N_1 + N_2 + \dots + N_k \subseteq M$ is an R-module isomorphism, when $\pi(a_1, a_2, \dots a_k) = a_1 + a_2 + \dots + a_k$)
2. $(N_1 + N_2 + \dots + N_{j-1} + N_{j+1} + \dots + N_k) \cap N_j = 0 \quad \forall j$
3. Each $z \in N_1 + N_2 + \dots + N_k$ has a unique expression $z = a_1 + a_2 + \dots + a_k$ for $a_i \in N_i$

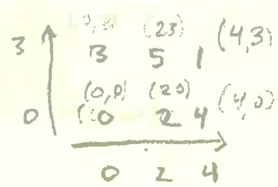
Example $\mathbb{Z}$-module, $M = \mathbb{Z}/6\mathbb{Z}$

$(4,3) + (2,0) = (0,3)$

$1 + 2 = 3$

$N_1 = \{0, 2, 4\} = \mathbb{Z}2 =$ "span (2)"

$N_2 = \{0, 3\} = \mathbb{Z}3 =$ "span (3)"



$N_1 + N_2 = \{0, 1, 2\ 3\ 4\ 5\} = M$

Since $N_1 \cap N_2 = 0 \quad N_1 \times N_2 \cong N_1 + N_2 = M$

$(x, y) \longmapsto x + y$

Note: This is not like a vector space in that $\text{span}(2) \neq \mathbb{Z}$, etc.

We next introduce the notion of free modules, for which such "one-dimensional" sub-modules are isomorphic to R.

<u>Definition</u> R-module F is <u>free</u> on $A \subseteq F$ if $\forall\, x \neq 0$ in F there exist unique non-zero $r_1\, r_2 \dots r_n \in R$ and $a_1\, a_2 \dots a_n \in A$ for which $x = \sum_{i=1}^{n} r_i a_i$. For commutative R, $|A|$ is called the <u>rank</u> of A.

<u>Example</u> $R^n = R \times R \times \dots \times R \quad A = \{(1,0,0 \dots 0), (0,1,0 \dots 0), (0,0,1 \dots 0), \dots (0,0,0, \dots 1)\}$

<u>Theorem 6</u> For any set A, there is ~~a~~ free R-module $F(A)$ on A. Moreover $F(A)$ satisfies the following universal property:

For any R-module M and any map $\varphi : A \to M$, there is a unique R-module homomorphism $\Phi : F(A) \to M$ ~~with $a = \varphi(a)\ \forall a \in A$~~.

$$\begin{tikzcd} A \arrow[r, "\text{inclusion}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\Phi"] \\ & M \end{tikzcd} \qquad \begin{array}{ccc} a & & \sum r_i a_i \\ \downarrow & & \downarrow \\ \varphi(a) & & \sum r_i\, \varphi(a_i) \end{array}$$

such that $\Phi(a) = \varphi(a)$ for all $a \in A$.

If A is <u>finite</u>, i.e. $A = \{a_1, a_2, \dots, a_n\}$ then $F(A) = Ra_1 \oplus Ra_2 \oplus \dots \oplus Ra_n \cong R^n$

$F(A) = \{ f \mid f \text{ is a fnct } A \to R \}$

$$\begin{tikzcd} A \arrow[r, "\text{incl}"] \arrow[dr, "\varphi"'] & F(A) \arrow[d, "\Phi"] \\ & M \end{tikzcd}$$