§7.7 Exponential Generating Functions

Now we introduce an entirely new kind of generating function – exponential generating functions. To motivate it, consider

$$P(n,k) = \frac{n!}{(n-k)!} = \left(\text{number of } k\text{-permutations of } n \text{ things}\right)$$

What if we wanted to work with the generating function for the sequence

$$P(n,0)\ P(n,1)\ P(n,2),\ \dots\ P(n,n)\ ?$$

The generating function is

$$g(x) = P(n,0) + P(n,1)\,x + P(n,2)\,x^2 + P(n,3)\,x^3 + \dots P(n,n)x^n$$

This simply does not simplify into a compact formula. That's a problem. To overcome it we try something new – replace each x^k with $x^k/k!$. Doing this gives what's called an exponential generating function.

$$g^{(e)}(x) = P(n,0) + P(n,1)\frac{x}{1!} + P(n\,2)\frac{x^2}{2!} + P(n,3)\frac{x^3}{3!} + \dots + P(n,n)\frac{x^n}{n!}$$

$$= \frac{P(n\,0)}{0!} + \frac{P(n,1)}{1!}x + \frac{P(n,2)}{2!}x^2 + \frac{P(n,3)}{3!}x^3 + \dots + \frac{P(n\,n)}{n!}x^n$$

$$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$$

$$= (1+x)^n$$

Dramatic simplifications like this one lead to our new definition

> **Definition** The exponential generating function for the sequence $h_0\ h_1\ h_2\ h_3\ \dots$ is
>
> $$g^{(e)}(x) = \sum_{k=0}^{\infty} h_k \frac{x^k}{k!} = h_0 + h_1 x + h_2\frac{x^2}{2!} + h_3\frac{x^3}{3!} + \dots$$

Our old kind of generating function is called an ordinary generating function.

Example Consider sequence 1, 1, 1, 1, 1, ...

Ordinary generating function for this sequence is

$$g(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

Exponential generating function for this sequence is

$$g^{(e)}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

Using exponential generating functions require these identities

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $e^{rx} = \sum_{k=0}^{\infty} \frac{(rx)^k}{k!} = \sum_{k=0}^{\infty} r^k \frac{x^k}{k!}$
- $\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$
- $\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

Also its sometimes handy to know tricks like this one:

$$\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots = e^x - 1 - x - \frac{x^2}{2} \quad \text{etc.}$$

Exponential generating functions can come to our rescue when ordinary generating functions fail us - particularly when dealing with permutations.

Motivating Example Find generating function for $h_0\ h_1\ h_2 \ldots$ where h_n is the number of n-permutations of $S = \{\underbrace{a, a, a}_{3}, \underbrace{b\ b\ b\ b \cdots}_{\infty}\}$

$$\overbrace{b\,b\,b \ldots b}^{n} \qquad a\,\overbrace{b\,b\,b \cdots b}^{n-1} \qquad a\,a\,\overbrace{b\,b \cdots b}^{n-2} \qquad a\,a\,a\,\overbrace{b\,b\,b \cdots b}^{n-3}$$

$$\binom{n}{0} \qquad \binom{n}{1} \text{ of these} \qquad \binom{n}{2} \text{ of these} \qquad \binom{n}{3} \text{ of these}$$

Thus $h_n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$

$$g^{(e)}(x) = \sum_{n=0}^{\infty}\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}\right]\frac{x^n}{n!}$$

$$1 + \left[\binom{1}{0}+\binom{1}{1}\right]x + \left[\binom{2}{0}+\binom{2}{1}+\binom{2}{2}\right]\frac{x^2}{2!} + \left[\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}\right]\frac{x^3}{3!} + \left[\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}\right]\frac{x^4}{4!} + \cdots$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{4!}\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)$$

<u>Reason</u>: Expanding this out, x^n term is

$$1 \cdot \frac{x^n}{n!} + x\,\frac{x^{n-1}}{(n-1)!} + \frac{x^2}{2!}\frac{x^{n-2}}{(n-2)!} + \frac{x^3}{3!}\frac{x^{n-3}}{(n-3)!}$$

$$= \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{2!(n-2)!} + \frac{1}{3!(n-3)!}\right)x^n$$

$$= \left(\frac{n!}{n!} + \frac{n!}{(n-1)!} + \frac{n!}{2!(n-2)!} + \frac{n!}{3!(n-3)!}\right)\frac{x^n}{n!}$$

$$= \left(\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}\right)\frac{x^n}{n!}$$

Here the first factor of $g(x)$ models the choices for the a's 0, 1, 2, or 3 of them. The second models the choices for the b's (i.e. any number of b's is allowable.

This illustrates our main theorem.

<u>Theorem 7.7.1</u> Suppose $S = \left\{\begin{matrix} a_1 & a_2 & a_3 & \cdots & a_k \\ n_1 & n_2 & n_3 & \cdots & n_k\end{matrix}\right\}$ is a multiset, and h_n is the number of n-permutations of S. Then the exponential generating function for $h_0\ h_1\ h_2\ h_3 \cdots$ is

$$g^{(e)}(x) = \left(\sum_{n=1}^{n_1}\frac{x^n}{n!}\right)\left(\sum_{n=1}^{n_2}\frac{x^n}{n!}\right)\cdots\left(\sum_{n=1}^{n_k}\frac{x^n}{n!}\right).$$

Here any n_i may be ∞ and any restrictions on the n_i (e.g. n_i must be even, etc.) are reflected in the terms $\frac{x^n}{n!}$ of the corresponding factor.

<u>Example</u> Let $S=\left\{\begin{matrix} a & b & c \\ 1 & \infty & \infty \end{matrix}\right\}$; and let h_n be the number of n-permutations of S, where there is at least one c. Find the exponential generating function for h_n and use it to find a formula for each h_n.

$$g^{(e)}(x) = (1+x)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)$$

$$= (1+x)\,e^x(e^x-1)$$

$$= (1+x)(e^{2x}-e^x)$$

$$= e^{2x}-e^x+xe^{2x}-xe^x$$

$$= \sum_{n=0}^{\infty}\frac{(2x)^n}{n!} - \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{K=0}^{\infty} x\frac{(2x)^n}{n!} - \sum_{K=0}^{\infty} x\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} 2^n\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty} n2^{n-1}\frac{x^n}{n!} - \sum_{n=0} n\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(2^n-1+n2^{n-1}-n\right)\frac{x^n}{n!}$$

Answer $\boxed{h_n = 2^n-1+n2^{n-1}-n}$

<u>Example</u> $S=\left\{\begin{matrix} a & b & c & d \\ \infty & \infty & \infty & \infty \end{matrix}\right\}$ Find generating function for h_n = (# of n-permutations with an even # of b's)

$$g^{(e)}(x) = \left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)$$

$$= (e^x)^3\,\tfrac{1}{2}(e^x+e^{-x}) = \tfrac{1}{2}e^{4x}+\tfrac{1}{2}e^{2x}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}4^n\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}2^n\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{4^n}{2}+\frac{2^n}{2}\right)\frac{x^n}{n!}$$

Thus $\boxed{h_n = \frac{4^n}{2}+\frac{2^n}{2} = 2^{2n-1}+2^{n-1}}$