Chapter 7 Recurrence Relations and Generating Functions

§7.1 Some Number Sequences

A recurrence relation is a way of defining a sequence of numbers. For now we will illustrate this with examples and delay the formal definition until Section 7.2.

Example A number sequence $h_0\ h_1\ h_2\ h_3 \cdots$ starts as

$$\underbrace{\overset{2}{\underset{h_0}{\uparrow}}\ \overset{0}{\underset{h_1}{\uparrow}}\ \overset{1}{\underset{h_2}{\uparrow}}}_{\text{initial values}}, \quad \text{and the } n^{th} \text{ term is } \underbrace{h_n = h_{n-1} + 3h_{n-2} + h_{n-3}}_{\text{recurrence relation.}}$$

The sequence is $2, 0, 1, 3, 6, 16, 37, 91, \cdots$

Question Is there a formula for h_n involving only n, and not $h_{n-1}\ h_{n-2}\ h_{n-3}$?
Later we will see how to answer this.

Example A sequence starts as $\underbrace{h_0 = 1}_{\text{initial value}}$, and the n^{th} term is $\underbrace{h_n = nh_{n-1}}_{\text{recurrence relation}}$

The sequence is

$$\underset{h_0}{\overset{1}{\uparrow}}\ \ \underset{h_1}{\overset{1}{\uparrow}}\ \ \underset{h_2}{\overset{2}{\uparrow}}\ \ \underset{h_3}{\overset{6}{\uparrow}}\ \ \underset{h_4}{\overset{24}{\uparrow}}\ \ \underset{h_5}{\overset{120}{\uparrow}} \cdots \underset{h_n}{\overset{n!}{\uparrow}} \cdots$$

Formula for n^{th} term is $h_n = n!$

Definitions Let a and q be numbers

The arithmetic sequence with initial value $h_0 = a$ and difference q is $a\ \ a+q\ \ a+2q\ \ a+3q\ \ a+4q \cdots$ defined with recurrence relation $h_n = h_{n-1} + q$.
Formula for n^{th} term is $h_n = a + nq$.

The geometric sequence with initial value $h_0 = a$ and ratio q is

$a\ \ aq\ \ aq^2\ \ aq^3\ \ aq^4 \cdots$ defined with recurrence relation $h_n = h_{n-1}q$
Formula for the n^{th} term is $h_n = aq^n$.

Definition The Fibonacci Sequence $f_0\ f_1\ f_2\ f_3\ f_4 \cdots$ has initial values $f_0 = 0$, $f_1 = 1$ and recurrence relation $f_n = f_{n-1} + f_{n-2}$

$0\ \ 1\ \ 1\ \ 2\ \ 3\ \ 5\ \ 8\ \ 13\ \ 21\ \ 34\ \ 55\ \ 89\ \ 144 \cdots$

Todays Goals

1. Find a formula for f_n (n^{th} term of Fibonacci Sequence) in terms of n.
2. Examine how this solves certain combinatorial problems.

Goal of § 7.2 (looking ahead):
Apply today's techniques to general formulas for other recurrence relations.

Goal 1 requires the following:

Lemma The Fibonacci Sequence $f_0\ f_1\ f_2\ f_3 \cdots$ obeys $f_{n+1}^2 - f_{n+1}f_n - f_n^2 = (-1)^n$.

Proof (Induction on n)

If $n=0$ we have $f_{n+1}^2 - f_{n+1}f_n - f_n^2 = f_1^2 - f_1 f_0 - f_0^2 = 1^2 - 1\cdot 0 - 0^2 = 1 = (-1)^0 = (-1)^n$.

Now assume this holds for $n-1$, that is, assume $f_n^2 - f_n f_{n-1} - f_{n-1}^2 = (-1)^{n-1}$.

Then
$$\begin{aligned} f_{n+1}^2 - f_{n+1}f_n - f_n^2 &= (f_n + f_{n-1})^2 - (f_n + f_{n-1})f_n - f_n^2 \\ &= f_n^2 + 2f_n f_{n-1} + f_{n-1}^2 - f_n^2 - f_{n-1}f_n - f_n^2 \\ &= -f_n^2 + f_n f_{n-1} + f_{n-1}^2 \\ &= (-1)(f_n^2 - f_n f_{n-1} - f_{n-1}^2) = (-1)(-1)^{n-1} = (-1)^n \quad \blacksquare \end{aligned}$$

Now divide $f_{n+1}^2 - f_{n+1}f_n - f_n^2 = (-1)^n$ by f_n^2. We get

$$\left(\frac{f_{n+1}}{f_n}\right)^2 - \frac{f_{n+1}}{f_n} - 1 = \frac{(-1)^n}{f_n^2}$$

Approaches 0 as $n \to \infty$

roots are $\dfrac{1 \pm \sqrt{1^2 - 4\cdot 1(-1)}}{2\cdot 1} = \dfrac{1 \pm \sqrt{5}}{2}$

Consequence: As $n \to \infty$, $\frac{f_{n+1}}{f_n}$ approaches a root of $x^2 - x - 1 = 0$

Conclusion: $\boxed{\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}}$

Spot check: 0 1 1 2 3 5 8 13 21 34 55 89 ...

($f_{10} = 55$, $f_{11} = 89$)

$$\left.\begin{aligned} \frac{f_{11}}{f_{10}} &= \frac{89}{55} = 1.6181\overline{8} \\ \frac{1+\sqrt{5}}{2} &= 1.6180339 \end{aligned}\right\} \text{agree to 3 decimal places.}$$

Now lets look at the roots of $x^2 - x - 1 = 0$ $\begin{cases} \varphi = \dfrac{1+\sqrt{5}}{2} \\ \lambda = \dfrac{1-\sqrt{5}}{2} \end{cases}$

$\varphi^2 - \varphi - 1 = 0 \Rightarrow \varphi^2 = \varphi + 1 \Rightarrow \varphi^n = \varphi^{n-2}\varphi^2 = \varphi^{n-2}(\varphi+1) = \varphi^{n-1} + \varphi^{n-2}$

$\lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda^2 = \lambda + 1 \Rightarrow \lambda^n = \cdots = \lambda^{n-1} + \lambda^{n-2}$

So we have $\boxed{\begin{aligned} \varphi^n &= \varphi^{n-1} + \varphi^{n-2} \\ \lambda^n &= \lambda^{n-1} + \lambda^{n-2} \end{aligned}}$

looks like $f_n = f_{n-1} + f_{n-2}$

Conclusion

$1\ \varphi\ \varphi^2\ \varphi^3\ \varphi^4\ \varphi^5 \dots$
$1\ \lambda\ \lambda^2\ \lambda^3\ \lambda^4\ \lambda^5 \dots$

Geometric sequences satisfying Fibonacci recurrence relation: $\varphi^n = \varphi^{n-1} + \varphi^{n-2}$ and $\lambda^n = \lambda^{n-1} + \lambda^{n-2}$.

That is, any term is the sum of the previous two terms.
However, these are <u>not</u> the Fibonacci sequence because the initial values are <u>not</u> $h_0 = 0$, $h_1 = 1$.

Notice that we can combine these two sequences.
Let $\alpha, \beta \in \mathbb{R}$ and define the sequence

$$\boxed{\alpha+\beta,\ \alpha\varphi+\beta\lambda,\ \alpha\varphi^2+\beta\lambda^2,\ \alpha\varphi^3+\beta\lambda^3,\ \alpha\varphi^4+\beta\lambda^4,\ \dots} \quad *$$

This also satisfies the Fibonacci recurrence relation:

$$\begin{aligned}
\alpha\varphi^n + \beta\varphi^n &= \alpha(\varphi^{n-1} + \varphi^{n-2}) + \beta(\lambda^{n-1} + \lambda^{n-2}) \\
&= (\alpha\varphi^{n-1} + \beta\lambda^{n-1}) + (\alpha\varphi^{n-2} + \beta\varphi^{n-2}) \\
&= [\text{sum of previous two terms}]
\end{aligned}$$

<u>Idea</u> choose α, β so first & second terms of $*$ are 0 & 1.
Then we will get the Fibonacci Sequence.

This means we need α and β to satisfy

$$\begin{cases} \alpha+\beta = 0 \\ \alpha\varphi+\beta\lambda = 1 \end{cases} \Rightarrow \begin{cases} \beta = -\alpha \\ \alpha\varphi - \alpha\lambda = 1 \end{cases} \Rightarrow \alpha(\varphi-\lambda) = 1 \Rightarrow \alpha\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$$

$$\Rightarrow \alpha\sqrt{5} = 1 \Rightarrow \boxed{\alpha = \frac{1}{\sqrt{5}} \quad \beta = -\frac{1}{\sqrt{5}}}$$

With this $*$ becomes the Fibonacci sequence itself.

$$\underset{\substack{\parallel \\ 0}}{\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}}},\ \underset{\substack{\parallel \\ 1}}{\frac{1}{\sqrt{5}}\varphi - \frac{1}{\sqrt{5}}\lambda},\ \underset{\substack{\parallel \\ 1}}{\frac{1}{\sqrt{5}}\varphi^2 - \frac{1}{\sqrt{5}}\lambda^2},\ \underset{\substack{\parallel \\ 2}}{\frac{1}{\sqrt{5}}\varphi^3 - \frac{1}{\sqrt{5}}\lambda^3},\ \underset{\substack{\parallel \\ 3}}{\frac{1}{\sqrt{5}}\varphi^4 - \frac{1}{\sqrt{5}}\lambda^4} \dots$$

<u>Theorem</u> The n^{th} term of the Fibanacci Sequence is

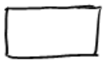
$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This is <u>Goal ①</u>, a formula for f_n.

Now let's begin to examine Goal (2) – how this type of thinking might solve certain combinatorial problems.

<u>Example</u> Given a $2 \times n$ grid, let h_n be the number of ways to cover it with dominoes. Find a formula for h_n.

2×1



$h_1 = 1$ way

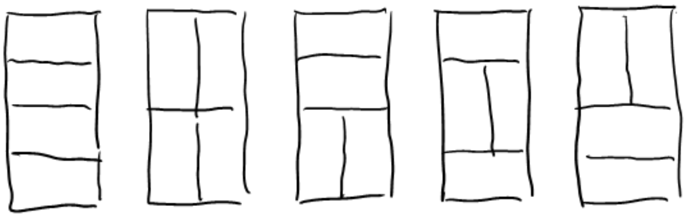
2×2

$h_2 = 2$ ways

2×3

$h_3 = 3$ ways

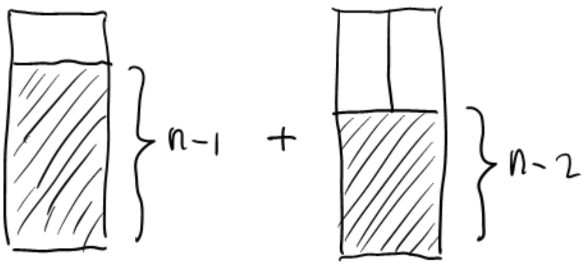
2×4



$h_4 = 5$ ways

$\vdots$

$2 \times n$



$h_n = h_{n-1} + h_{n-2}$

(h_{n-1} ways) (h_{n-2} ways)

<u>Answer</u>: A $2 \times n$ grid can be covered with dominoes in $h_n = f_{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ ways.

In this chapter we will often be confronted with a problem such as this. Given some combinatorial question, we will model it with a sequence defined by a recurrence relation catered to the problem and <u>then</u> use techniques like today's to find a general formula for the n^{th} term.

[Note: The sequence will <u>not</u> always be the Fibonacci Sequence]