CHAPTER 8

Limits and Limit Laws

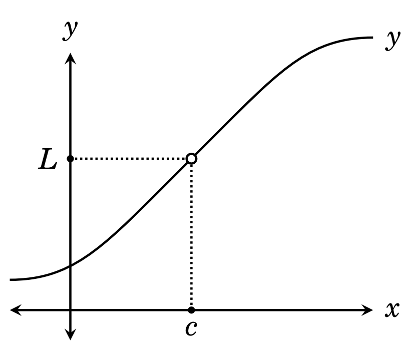
Chapter 7 showed that finding slopes of tangent lines entails a new concept called a *limit*. As finding slope of tangents is one of the central problems of calculus, it is necessary to undertake a thorough study of limits. That is the purpose of Part 2 of this text, where we concentrate purely on limits and postpone tangent slopes until Part 3.

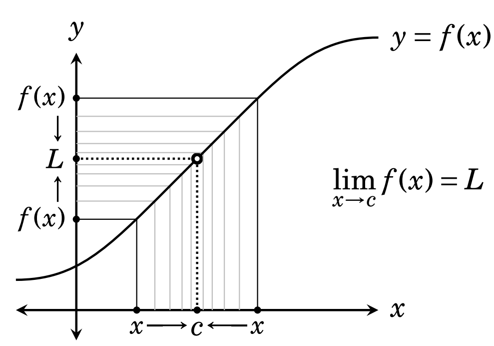
8.1 Definitions and Examples

Here is an intuitive working definition of a limit. (A more robust definition comes later, but this one is sufficient for a first course in calculus.)

> **Definition 8.1 (Informal definition of a limit)**
> Suppose f is a function and c is a number. Then $\lim_{x\to c} f(x)$ is the number that $f(x)$ approaches as x approaches c

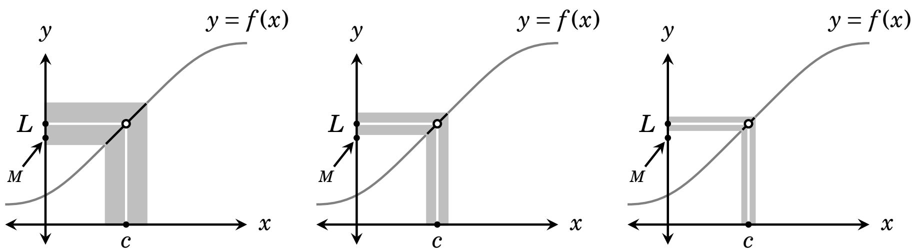
We pronounce $\lim_{x\to c} f(x)$ as "*the limit as x approaches c of $f(x)$.*" As an example, consider the function f graphed below on the left. It happens that $f(c)$ is not defined, as indicated by the hole in the graph over c. But the hole has y-coordinate L. For any $x \neq c$, the corresponding value $f(x)$ is either greater than L or less than L. But the closer x is to c, the closer $f(x)$ is to L, as illustrated on the right. We express this by saying $\lim_{x\to c} f(x) = L$.





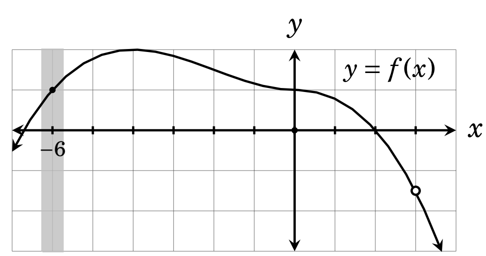
This situation is typical. Often a function value $f(c)$ is not defined, but we still need to know how f "behaves" when x is near c. The limit $\lim_{x\to c} f(x) = L$ tells the story.

Here is another way to envision $\lim_{x\to c} f(x) = L$. Draw a narrow vertical band around $x = c$, intersecting the x-axis at values that are close to c. Then draw a horizontal band meeting the part of the graph of f that intersects the vertical band. The horizontal band meets the y-axis at all the points $f(x)$ for which x is in the vertical band. Drawing the vertical band narrower forces the horizontal band to become narrower too, squeezing in on a number L. Forcing x closer to c forces $f(x)$ closer to L. The number L is $\lim_{x\to c} f(x)$.



Notice that L is *the exact value* that $f(x)$ gets close to as x gets closer to c. If we take *any* other number $M \neq L$, then we could force $f(x)$ closer to L than to M by making the vertical band sufficiently narrow, as shown above.

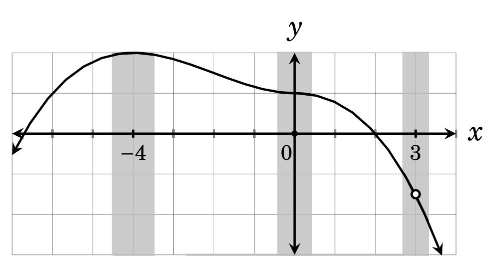
Example 8.1 For the function $f(x)$ on the right, find $\lim_{x\to -6} f(x)$, $\lim_{x\to -4} f(x)$, $\lim_{x\to 0} f(x)$ and $\lim_{x\to 3} f(x)$.



For $\lim_{x\to -6} f(x)$, focus on the graph near $x = -6$, in the narrow shaded band.

In this band we can see that as x approaches -6, the value $f(x)$ approaches 1. Therefore $\lim_{x\to -6} f(x) = 1$.

To find the limit as x approaches -4, focus on a narrow strip near -4. Looking at this strip (and ignoring the rest of the graph) we see that as x approaches -4, $f(x)$ approaches 2. Thus $\lim_{x\to -4} f(x) = 2$.



Turning to $\lim_{x\to 0} f(x)$, concentrate on the graph in a narrow strip around $x = 0$. There we see $f(x)$ approaching 1 as x approaches 0, so $\lim_{x\to 0} f(x) = 1$. Finally, the graph in a narrow strip around $x = 3$ shows $\lim_{x\to 3} f(x) = -1.5$. ✍

Example 8.2 Let $g(x) = \dfrac{x^2-1}{x-1}$. Note $g(1) = \frac{0}{0}$ is undefined. Find $\lim\limits_{x \to 1} g(x)$.

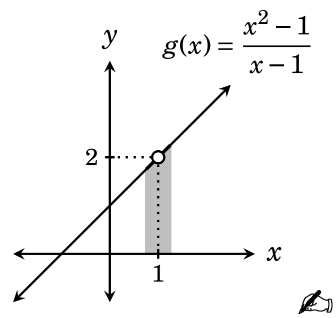
We are in a situation where $x = 1$ is a "bad point" for the function $g(x)$, and the limit is asking what happens to $g(x)$ as x approaches the forbidden 1. The table below (done with a calculator) suggests that as x approaches 1, the quantity $g(x) = \frac{x^2-1}{x-1}$ approachs 2. We surmise $\lim\limits_{x \to 1} g(x) = 2$.

x	0	0.9	0.99	0.999	0.9999	→	1	←	1.0001	1.001	1.01	1.1	2
$\dfrac{x^2-1}{x-1}$	1	1.9	1.99	1.999	1.9999	→	?	←	2.0001	2.001	2.01	2.1	3

We graphed $g(x)$ on page 15. Recall that

$$g(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1,$$

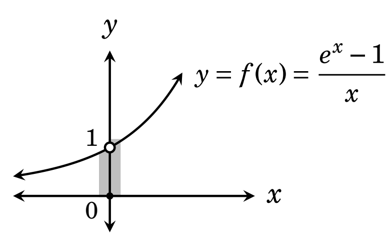
so $g(x) = x+1$ whenever $x \neq 1$. Its graph is thus a line of slope 1 and y-intercept 1, and a hole at $(1,2)$, as $g(1)$ is not defined. From the graph we see that $\lim\limits_{x \to 1} g(x) = 2$.



Example 8.3 Consider $f(x) = \dfrac{e^x-1}{x}$. Notice $f(0) = \dfrac{0}{0}$ is undefined. But let's see what happens to $f(x)$ as x gets close to this "bad" point $x = 0$. That is, let's investigate $\lim\limits_{x \to 0} f(x)$. The following table tallies $f(x)$ for x close to 0.

x	−1	−0.1	−0.01	−0.001	→	0	←	0.001	0.01	0.1	1
$\dfrac{e^x-1}{x}$	0.623	0.951	0.995	0.999	→	*	←	1.001	1.005	1.051	1.718

The table shows that $f(x)$ appears to get progressively closer to 1 the closer and closer x gets to 0. From this we would expect $\lim\limits_{x \to 0} f(x) = 1$. This is supported by the graph of $f(x)$ on the right, done with a graphing utility. The hole in the graph indicates that $f(0)$ is not defined.

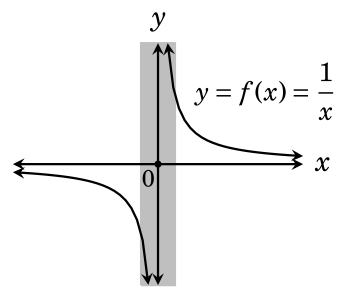


But we do see that for values x close to 0 (like those inside the shaded band) the corresponding value $f(x)$ is close to 1, and $f(x)$ seems closer to 1 the closer x is to 0. Again we surmise $\lim\limits_{x \to 0} f(x) = 1$. But this is just a guess based on numeric and graphic evidence. Later we will develop rigorous techniques that show that this limit is indeed 1.

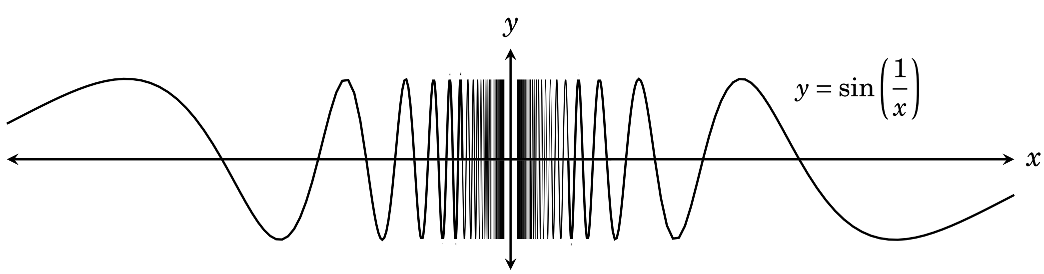
Some limits simply make no sense, and it is important to know how to recognize and handle this. This page shows a few examples. We begin with $\lim_{x\to 0} \frac{1}{x}$. Below we tally $\frac{1}{x}$ for some x values close to 0. Notice how $\frac{1}{x}$ becomes a bigger and bigger number (either positive or negative) the closer we let x to 0.

x	-1	-0.1	-0.01	-0.001	$\longrightarrow$	0	$\longleftarrow$	0.001	0.01	0.1	1
$\frac{1}{x}$	-1	-10	-100	-1000	$\longrightarrow$	$?$	$\longleftarrow$	1000	100	10	1

We see this behavior in the graph of $f(x) = \frac{1}{x}$. Look at the shaded band for x values close to 0. Here $f(x) = \frac{1}{x}$ goes to ∞ as x approaches 0 from the right. If x approaches 0 from the left, then $f(x)$ goes toward $-\infty$. As $f(x)$ approaches no definite value as x approaches 0, we say that $\lim_{x\to 0} f(x)$ **does not exist**, abbreviated **DNE**. Thus $\lim_{x\to 0} \frac{1}{x}$ DNE.



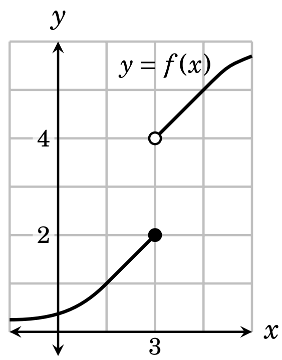
Another way that $\lim_{x\to c} f(x)$ will not exist is if $f(x)$ oscillates infinitely often as x approaches c. This is the case for the function $\sin\left(\frac{1}{x}\right)$ graphed below. As x approaches 0, the number $\frac{1}{x}$ plugged into sin grows bigger and bigger, approaching infinity, so $\sin\left(\frac{1}{x}\right)$ just bounces up and down, faster and faster the closer x gets to 0. (We even gave up on the part of the graph where x is close to 0 because it would look like a solid region of black ink.) We say $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ DNE (Does Not Exist).



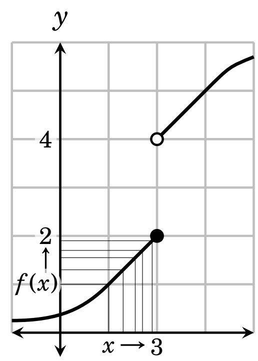
For the most part, the limits that will be most important will exist, and can be found. Therefore we regard the examples on this page as somewhat pathological. You need not dwell on them – their only purpose is to underscore the fact that it is possible that certain limits do not exist.

8.2 Right- and Left-Hand Limits

There is another way a limit can fail to exist, and this scenario leads to the ideas of *right- and left-hand limits*, defined at the bottom of this page. As motivation, consider the following situation.



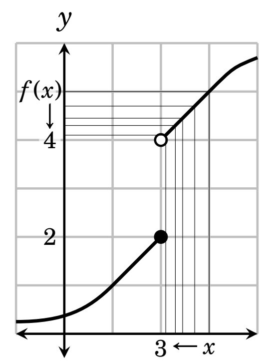
Above is a function $f(x)$ that makes an abrupt jump at $x = 3$. What happens to $f(x)$ as x approaches 3? The answer depends on which direction x approaches 3 from.



If x approaches 3 *from the left*, then $f(x)$ approaches 2. We express this as

$$\lim_{x \to 3^-} f(x) = 2,$$

where $x \to 3^-$ means x is to the *left* of 3.



If x approaches 3 *from the right*, then $f(x)$ approaches 4. We express this as

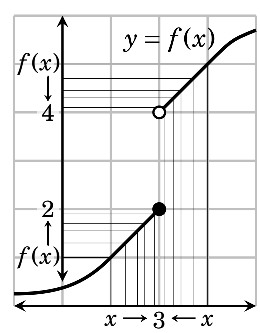
$$\lim_{x \to 3^+} f(x) = 4,$$

where $x \to 3^+$ means x is to the *right* of 3.

The two scenarios are combined on the right. Notice that x approaching 3 does not force $f(x)$ to approach a single number. As $f(x)$ approaches different numbers depending on how x approaches 3 we have to say $\lim_{x \to 3} f(x)$ DNE.

But we do have $\lim_{x \to 3^-} f(x) = 2$ and $\lim_{x \to 3^+} f(x) = 4$.

This example motivates a general definition.



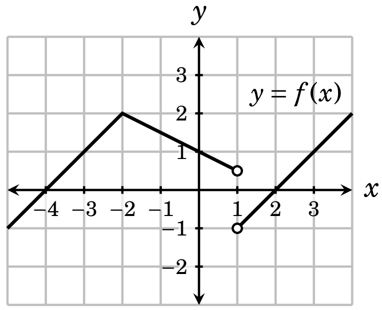
Definition 8.2 Right- and left-hand limits:

- $\lim_{x \to c^+} f(x)$ is the number $f(x)$ approaches as x approaches c from the right.
- $\lim_{x \to c^-} f(x)$ is the number $f(x)$ approaches as x approaches c from the left.

As noted above, if $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then $\lim_{x \to c} f(x)$ DNE. But if both the right- and left-hand limits *are equal*, then $\lim_{x \to c} f(x)$ equals this number.

Fact 8.1 $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x) \iff \lim_{x \to c} f(x) = L$

Example 8.4 A function f is graphed. Be sure you agree with the limits.



$\lim_{x\to -4} f(x) = 0$ $\quad\lim_{x\to -3} f(x) = 1$ $\quad\lim_{x\to 1} f(x)$ DNE

$\lim_{x\to -2} f(x) = 2$ $\quad\lim_{x\to 3^+} f(x) = 1$ $\quad\lim_{x\to 1^-} f(x) = 0.5$

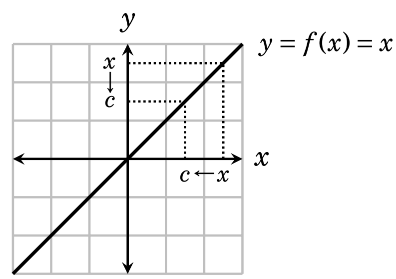
$\lim_{x\to 0} f(x) = 1$ $\quad\lim_{x\to 3} f(x) = 1$ $\quad\lim_{x\to 1^+} f(x) = -1$

✍

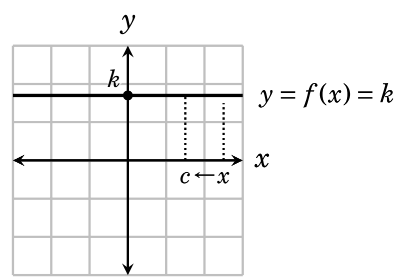
8.3 Limit Laws

We have computed limits visually (from graphs) and numerically (from tables). Both approaches are vital for a proper understanding of limits. Now we come to a third approach: algebraically. We will develop rules that compute limits quickly without reference to graphs or numeric evidence.

We begin with two of the simplest limits imaginable. Consider $f(x) = x$, the linear function $y = 1 \cdot x + 0$, whose graph is a straight line with slope 1 and y intercept 0. Clearly, for any number c we have $\lim_{x\to c} f(x) = c$. Let's write this as $\lim_{x\to c} x = c$.



Next, let k be a number and consider the function $f(x) = k$, which is a linear function $f(x) = 0 \cdot x + k$. Its graph is a horizontal line of slope 0 and height k. As x approaches c, the value $f(x)$ doesn't exactly approach k – it's already there. Thus $\lim_{x\to c} f(x) = k$. We write this as $\lim_{x\to c} k = k$.



Further reasoning from familiar graphs suggests $\lim_{x\to c} x^n = c^n$, and $\lim_{x\to c} a^x = a^c$ (for a positive base a) . Here is a summary.

Fact 8.2 Simple Limits Laws

a. $\lim_{x\to c} x^n = c^n$ $\qquad$ b. $\lim_{x\to c} x = c$

c. $\lim_{x\to c} k = k$ (for k constant) $\qquad$ d. $\lim_{x\to c} a^x = a^c$ (for $a > 0$)

Regard these as simple (and obvious) rules-for evaluating limits. Examples: $\lim_{x\to 3} x = 3$, $\lim_{x\to 3} x^2 = 3^2 = 9$, $\lim_{x\to \sqrt{2}} x = \sqrt{2}$, $\lim_{x\to 4} 7 = 7$, $\lim_{x\to \pi} \sqrt{2} = \sqrt{2}$, $\lim_{x\to 3} 2^x = 2^3 = 8$.

Sometimes a limit problem can be reduced to two or more simpler limits. For example, it is reasonable that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

In words, this says that as x approaches c, the value $\frac{f(x)}{g(x)}$ approaches what $f(x)$ approaches divided by what $g(x)$ approaches. In practical terms, it reduces the limit on the left to two simpler limits on the right.

This vein yields seven laws. They work for left- and right-hand limits too.

Limit Laws Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. Then:

1. $\lim_{x\to c} \big(f(x)+g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. $\lim_{x\to c} \big(f(x)-g(x)\big) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$ .... where k is a constant
4. $\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$
5. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ .... provided $\lim_{x\to c} g(x) \neq 0$
6. $\lim_{x\to c} \big(f(x)\big)^n = \left(\lim_{x\to c} f(x)\right)^n$ .... for an integer $n > 0$
7. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ ..... for an integer $n > 0$, and $\lim_{x\to c} f(x) > 0$ if n is even

Often you can evaluate (i.e., compute) a limit by using a combination of the above limit laws. For example, consider evaluating $\lim_{x\to 3}\left(x^2+5x-4\right)$. Using laws a, b, c (page 128) and 1–7 above, we get

$$\begin{aligned}
\lim_{x\to 3}\left(x^2+5x-4\right) &= \lim_{x\to 3}(x^2+5x) - \lim_{x\to 3} 4 && \text{(Law 2)}\\
&= \lim_{x\to 3}(x^2+5x) - 4 && \text{(Law c)}\\
&= \lim_{x\to 3} x^2 + \lim_{x\to 3} 5x - 4 && \text{(Law 1)}\\
&= \lim_{x\to 3} x^2 + 5\lim_{x\to 3} x - 4 && \text{(Law 3)}\\
&= 3^2 + 5\cdot 3 - 4 = \mathbf{20} && \text{(Laws a,b,c)}
\end{aligned}$$

That was a lot of work for not much payoff. After a little practice we learn to do this all in one step, applying the rules mentally:

$$\lim_{x\to 3}\left(x^2+5x-4\right) = 3^2+5\cdot 3-4 = 20.$$

For another example, note that with the help of Law 7 we get

$$\lim_{x\to 5}\frac{x^3+x}{x-1} = \frac{\lim\limits_{x\to 5}(x^3+x)}{\lim\limits_{x\to 5}(x-1)} = \frac{5^3+5}{5-1} = \frac{130}{4} = \frac{65}{2}.$$

Using the limit laws as we did in the previous two problems leads to two facts. To solve a limit of a polynomial or a rational function, just plug the number being approached into the function.

> **Fact 8.3** (Limits of polynomial and rational functions)
>
> 1. If $p(x)$ is a polynomial, then $\lim\limits_{x\to c} p(x) = p(c)$.
> 2. If $\dfrac{p(x)}{q(x)}$ is a rational function, then $\lim\limits_{x\to c}\dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$. (Provided $q(c)\neq 0$.)

Examples: $\lim\limits_{x\to\sqrt{2}}\left(x^4+x^2+1\right) = \sqrt{2}^4+\sqrt{2}^2+1 = 7$ and $\lim\limits_{x\to -2}\dfrac{x^2+1}{x-1} = \dfrac{(-2)^2+1}{-2-1} = -\dfrac{5}{3}$.

We have been using x as the variable in our limits, but any reasonable letter or symbol would work. Thus $\lim\limits_{h\to\frac{1}{2}}\left(h^2-h+3\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}+3 = \frac{1}{4}-\frac{1}{2}+3 = \frac{11}{4}$.

Example 8.5 Of course not every limit will involve a polynomial or rational function, but still our limit laws may apply.

$$\begin{aligned}
\lim_{x\to 2}\sqrt{x^2+1}\left(\frac{x+1}{x-1}\right)^5 &= \lim_{x\to 2}\sqrt{x^2+1}\cdot\lim_{x\to 2}\left(\frac{x+1}{x-1}\right)^5 && \text{(Law 4)}\\
&= \sqrt{\lim_{x\to 2}\left(x^2+1\right)}\cdot\left(\lim_{x\to 2}\frac{x+1}{x-1}\right)^5 && \text{(Laws 7, 6)}\\
&= \sqrt{2^2+1}\cdot\left(\frac{2+1}{2-1}\right)^5\\
&= \sqrt{5}\cdot 3^5 = 243\sqrt{5}
\end{aligned}$$

✍

Example 8.6 Find $\lim\limits_{x\to 3}\left(x^3-4x-7\right)^{\frac{2}{3}}$.

Rewriting this as a radical and using limit laws 6 and 7, we get

$$\begin{aligned}
\lim_{x\to 3}\left(x^3-4x-7\right)^{\frac{2}{3}} &= \lim_{x\to 3}\sqrt[3]{x^3-4x-7}^{\,2} = \left(\lim_{x\to 3}\sqrt[3]{x^3-4x-7}\right)^2\\
&= \sqrt[3]{\lim_{x\to 3}(x^3-4x-7)}^{\,2} = \sqrt[3]{8}^{\,2} = 4.
\end{aligned}$$

Note that limit laws 6 and 7 together imply $\lim\limits_{x\to c}\left(f(x)\right)^{\frac{p}{q}} = \left(\lim\limits_{x\to c} f(x)\right)^{\frac{p}{q}}$. ✍

Example 8.7 Investigate $\lim_{z \to 2} \left(\sqrt{z-2} + 1 \right)$.

Here we have to be careful. If z approaches 2 from the left (e.g. $z = 1.999$, etc.) then the quantity $z-2$ inside the radical is negative, and its square root is not defined. Therefore $\lim_{z \to 2^-} \left(\sqrt{z-2} + 1 \right)$ DNE. But if z is to the right of 2, $z-1$ is positive, which causes no problem with the radical. Limit laws give

$$\lim_{z \to 2^+} \left(\sqrt{z-2} + 1 \right) = \lim_{z \to 2^+} \sqrt{z-2} + \lim_{z \to 2^+} 1 = \sqrt{\lim_{z \to 2^+} (z-2)} + 1 = \sqrt{2-2} + 1 = 1.$$

But as the left-hand limit doesn't exist, we must say $\lim_{z \to 2} \left(\sqrt{z-2} + 1 \right)$ DNE. ✍

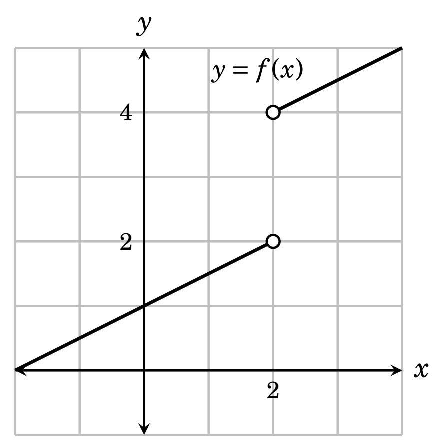
Example 8.8 Suppose $f(x) = \frac{x}{2} + \frac{|x-2]}{x-2} + 2$. Investigate $\lim_{x \to 2} f(x)$.

Notice that $f(2)$ is not defined because it involves division by zero. Moreover, $f(x)$ behaves differently depending on whether x is to the right or left of 2.

If $x > 2$, then $x-2$ is positive, so $|x-2| = x-2$ and $\frac{|x-2|}{x-2} = 1$, and $f(x) = \frac{x}{2} + 3$.
If $x < 2$, then $x-2$ is negative, so $|x-2| = -(x-2)$ and $\frac{|x-2|}{x-2} = -1$, so $f(x) = \frac{x}{2} + 1$.

Therefore $f(x)$ is a piecewise function $f(x) = \begin{cases} \frac{1}{2}x + 3 & \text{if } x > 2 \\ \frac{1}{2}x + 1 & \text{if } x < 2 \end{cases}$.

Because $f(x)$ uses a different rule depending on whether x is to the right or left of 2, we will investigate the right and left-hand limits separately. Observe that $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{1}{2}x + 3 \right) = \frac{1}{2}2 + 3 = 4$. On the other hand, we have $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{1}{2}x + 1 \right) = \frac{1}{2}2 + 1 = 2$. Because the right- and left-hand limits are different, we conclude $\lim_{x \to 2} f(x)$ DNE.



Our answers make sense when we look at the graph of f. It comes in two parts, each easy to sketch. Cover and ignore the part of the grid to the right of $x = 2$, and concentrate on the part to the left. There we have $f(x) = \frac{1}{2}x + 1$, whose graph is a line with slope $\frac{1}{2}$ and y-intercept 1, as drawn. Now ignore the part of the grid to *the left* of $x = 2$, and focus on the part to *the right*, where $f(x) = \frac{1}{2}x + 3$. Here the graph is a line with slope $\frac{1}{2}$ and y-intercept 3. We draw only the part of this line to the right of $x = 2$. ✍

Exercises for Chapter 8

1. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(1) =$

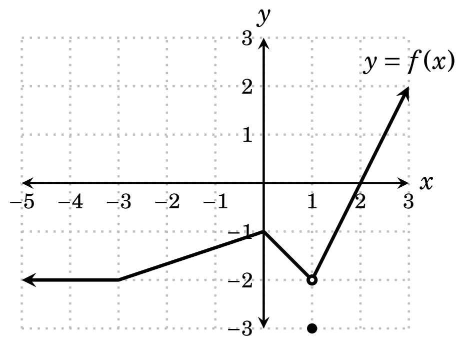
(b) $\lim\limits_{x\to 1} f(x) =$

(c) $\lim\limits_{x\to 1} f(2x) =$

(d) $f \circ f(2) =$

(e) $\lim\limits_{x\to -3} f(x) =$

(f) $\lim\limits_{x\to -3} \dfrac{f(x)}{x^2} =$



2. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(1) =$

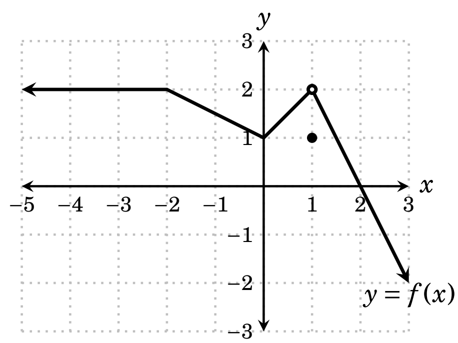
(b) $\lim\limits_{x\to 1} f(x) =$

(c) $\lim\limits_{x\to 1} f(-3x) =$

(d) $f \circ f(2) =$

(e) $\lim\limits_{x\to 0} f(x) =$

(f) $\lim\limits_{x\to 0} \dfrac{5f(x)}{x+3} =$



3. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(-1) =$

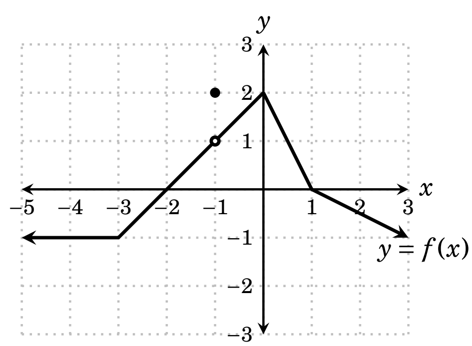
(b) $\lim\limits_{x\to -1} f(x) =$

(c) $\lim\limits_{x\to 2} f(x) =$

(d) $\lim\limits_{x\to 1} f(-2x) =$

(e) $f \circ f(1) =$

(f) $\lim\limits_{x\to 1} \dfrac{1+f(x)}{1+x} =$



4. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(-1) =$

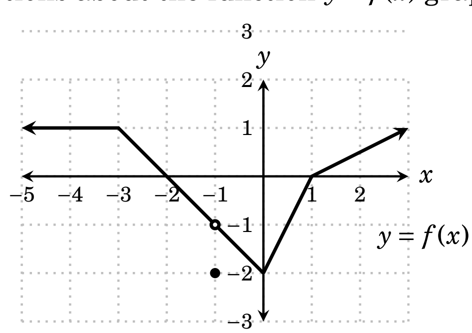
(b) $\lim\limits_{x\to -1} f(x) =$

(c) $\lim\limits_{x\to -3} f(x) =$

(d) $\lim\limits_{x\to 1} f(2x) =$

(e) $f \circ f(3) =$

(f) $\lim\limits_{x\to 2} x^2 f(x) =$



5. Answer the following questions about the function $y = f(x)$ graphed below.

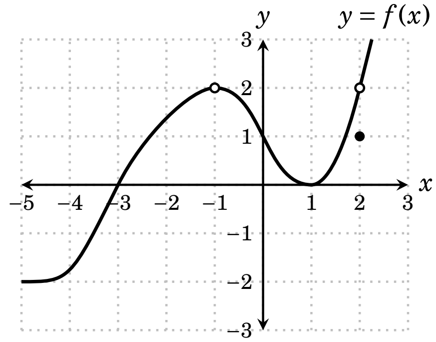
(a) $f(2) =$

(b) $\lim_{x\to 2} f(x) =$

(c) $f(-3) =$

(d) $\lim_{x\to -3} f(x) =$

(e) $\lim_{x\to -1} \dfrac{f(x)+1}{2f(x)} =$



6. Answer the following questions about the function $y = f(x)$ graphed below.

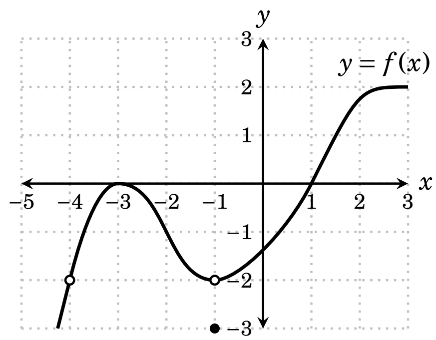
(a) $f(-3) =$

(b) $\lim_{x\to -3} f(x) =$

(c) $f(-1) =$

(d) $\lim_{x\to -1} f(x) =$

(e) $\lim_{x\to -1} \sqrt{1-2f(x)} =$



7. Answer the following questions about the function $y = f(x)$ graphed below.

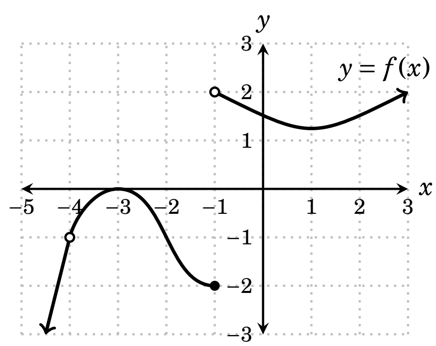
(a) $f(-1) =$

(b) $\lim_{x\to -1^+} f(x) =$

(c) $\lim_{x\to -1^-} f(x) =$

(d) $\lim_{x\to -1} f(x) =$

(e) $\lim_{x\to -4^+} f(x) =$



8. Answer the following questions about the function $y = f(x)$ graphed below.

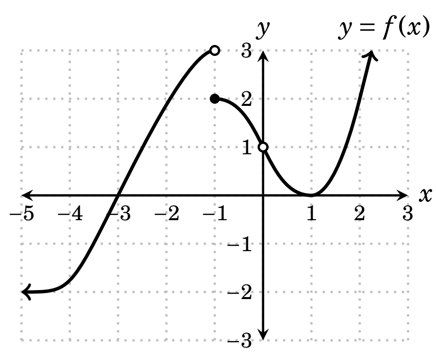
(a) $f(-1) =$

(b) $\lim_{x\to -1^+} f(x) =$

(c) $\lim_{x\to -1^-} f(x) =$

(d) $\lim_{x\to -1} f(x) =$

(e) $\lim_{x\to 0^+} f(x) =$



9. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(1) =$

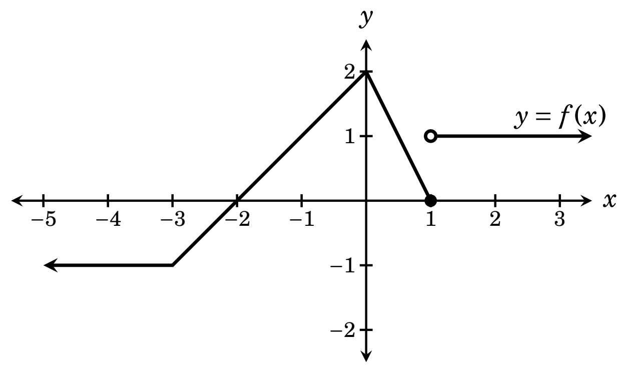
(b) $f \circ f(2) =$

(c) $\lim_{x \to 0} f(x) =$

(d) $\lim_{x \to 1} f(x) =$

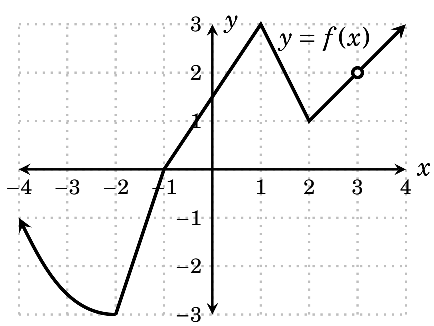
(e) $\lim_{x \to 1^+} f(x) =$

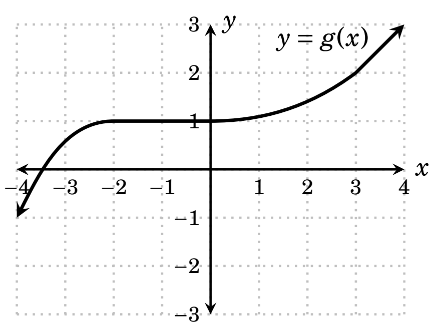
(f) $\lim_{x \to 1^-} f(x) =$



10. Given the two functions $f(x)$ and $g(x)$ graphed below, find the following values

(a) $\lim_{x \to 3} f(x)$ (b) $\lim_{x \to -2} \dfrac{3f(x)g(x)}{\sqrt{12 + f(x)}}$ (c) $g \circ f(-2)$ (d) $\lim_{x \to 3} f(g(x))$





For the remaining exercises, evaluate (i.e., compute) the limits.

11. $\lim_{x \to -1} \left(3x^2 - 4x + 1\right)$

12. $\lim_{x \to 1/2} \left(x^3 - x\right)$

13. $\lim_{x \to 0} \left(x^3 + x^2 + x + 1\right)$

14. $\lim_{x \to 5} \left(x^2 - 5x\right)$

15. $\lim_{x \to 1} \dfrac{x^2 - 5x}{x - 3}$

16. $\lim_{x \to 0} \dfrac{x^2 - 5x}{x - 3}$

17. $\lim_{x \to -2} \dfrac{x^2 + 7x}{x^2 - x + 3}$

18. $\lim_{x \to 2} 3^x \left(x^2 + 10\right)$

19. $\lim_{x \to 0} e^x \sqrt{x^5 + 100}$

20. $\lim_{x \to 2} \dfrac{x3^x}{x - 3}$

21. $\lim_{x \to 3} \left(\dfrac{6}{x} + \dfrac{\sqrt{x+1}}{x+3}\right)$

22. $\lim_{x \to -4} \sqrt{\dfrac{x}{x - 3}}$

23. $\lim_{x \to 5} \dfrac{1}{5x^2 - 3x + 5}$

24. $\lim_{x \to 2} \left(\dfrac{1}{4} + \dfrac{8}{x^2}\right)^{\frac{3}{2}}$

25. $\lim_{x \to 3} \left(\dfrac{x^2 - 1}{x^3}\right)^{\frac{2}{3}}$

Solutions for Chapter 8

1. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(1) = -3$

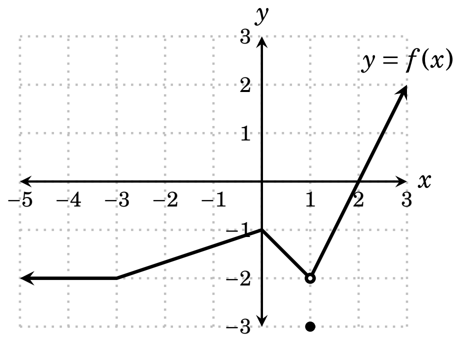
(b) $\lim_{x\to 1} f(x) = -2$

(c) $\lim_{x\to 1} f(2x) = 0$

(d) $f \circ f(2) = -1$

(e) $\lim_{x\to -3} f(x) = -2$

(f) $\lim_{x\to -3} \dfrac{f(x)}{x^2} = -\dfrac{2}{9}$



3. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(-1) = 2$

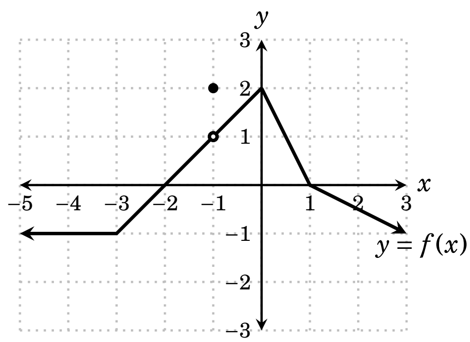
(b) $\lim_{x\to -1} f(x) = 1$

(c) $\lim_{x\to 2} f(x) = -0.5$

(d) $\lim_{x\to 1} f(-2x) = 0$

(e) $f \circ f(1) = 2$

(f) $\lim_{x\to 1} \dfrac{1+f(x)}{1+x} = \dfrac{1}{2}$



5. Answer the following questions about the function $y = f(x)$ graphed below.

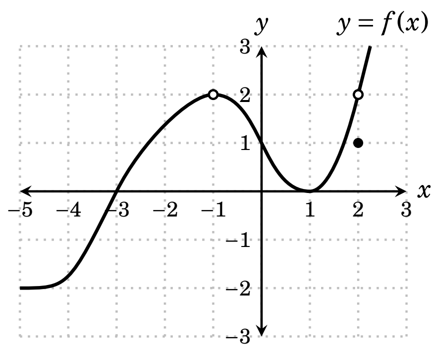
(a) $f(2) = 1$

(b) $\lim_{x\to 2} f(x) = 2$

(c) $f(-3) = 0$

(d) $\lim_{x\to -3} f(x) = 0$

(e) $\lim_{x\to -1} \dfrac{f(x)+1}{2f(x)} = \dfrac{3}{4}$



7. Answer the following questions about the function $y = f(x)$ graphed below.

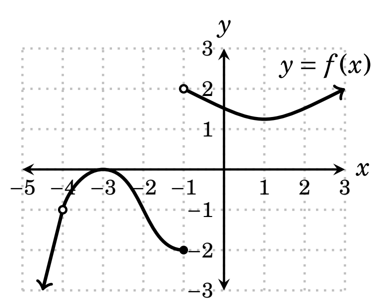
(a) $f(-1) = -2$

(b) $\lim_{x\to -1^+} f(x) = 2$

(c) $\lim_{x\to -1^-} f(x) = -2$

(d) $\lim_{x\to -1} f(x) = \text{DNE}$

(e) $\lim_{x\to -4^+} f(x) = -1$



9. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $f(1) = 0$

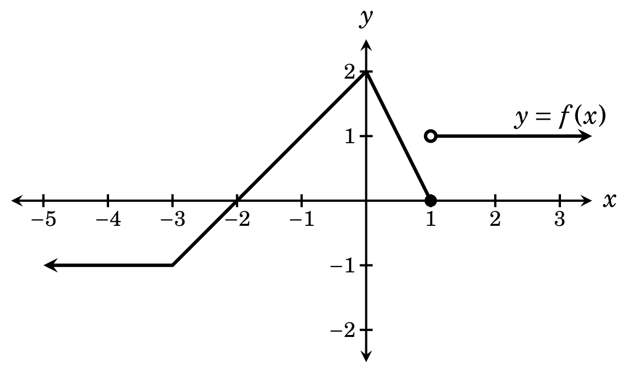
(b) $f \circ f(2) = 0$

(c) $\lim_{x\to 0} f(x) = 2$

(d) $\lim_{x\to 1} f(x) = \text{DNE}$

(e) $\lim_{x\to 1^+} f(x) = 1$

(f) $\lim_{x\to 1^-} f(x) = 0$



11. $\lim_{x\to -1}\left(3x^2 - 4x + 1\right) = 3(-1)^2 - 4(-1) + 1 = 8$

13. $\lim_{x\to 0}\left(x^3 + x^2 + x + 1\right) = 0^3 + 0^2 + 0 + 1 = 1$

15. $\lim_{x\to 1} \frac{x^2 - 5x}{x - 3} = \frac{1^2 - 5\cdot 1}{1 - 3} = 2$

17. $\lim_{x\to -2} \frac{x^2 + 7x}{x^2 - x + 3} = \frac{(-2)^2 + 7\cdot(-2)}{(-2)^2 - (-2) + 3} = -\frac{10}{9}$

19. $\lim_{x\to 0} e^x\sqrt{x^5 + 100} = \left(\lim_{x\to 0} e^x\right)\left(\lim_{x\to 0}\sqrt{x^5 + 100}\right) = e^0\sqrt{\lim_{x\to 0}(x^5 + 100)} = 1\cdot\sqrt{0 + 100} = 10$

21. $\lim_{x\to 3}\left(\frac{6}{x} + \frac{\sqrt{x+1}}{x+3}\right) = \lim_{x\to 3}\left(\frac{6}{x}\right) + \lim_{x\to 3}\left(\frac{\sqrt{x+1}}{x+3}\right) = \frac{6}{3} + \frac{\sqrt{3+1}}{3+3} = 2 + \frac{2}{6} = \frac{7}{3}$

23. $\lim_{x\to 5}\frac{1}{5x^2 - 3x + 5} = \frac{1}{5\cdot 5^2 - 3\cdot 5 + 5} = \frac{1}{115}$

25. $\lim_{x\to 3}\left(\frac{x^2 - 1}{x^3}\right)^{\frac{2}{3}} = \left(\lim_{x\to 3}\frac{x^2 - 1}{x^3}\right)^{\frac{2}{3}} = \left(\frac{3^2 - 1}{3^3}\right)^{\frac{2}{3}} = \left(\frac{8}{27}\right)^{\frac{2}{3}} = \sqrt[3]{\frac{8}{27}}^{\,2} = \frac{4}{9}$