CHAPTER 44

Integration by Substitution

Over the past five chapters we have seen that the process of finding indefinite integrals (that is, the process of *integration*) is essential in calculus. For this, we have so far relied on a relatively sparse list of integration rules that come from reversing differentiation rules.

Here is a summary of our integration rules so far. The only difference between this list and the one on page 424 is that here the variable x has been replaced by u. (For a reason that will soon become apparent.)

Integration Rules

$\int c\,du$	$=$	$cu+C$	$\int \sec^2(u)\,du$	$=$	$\tan(u)+C$
$\int u^n\,du$	$=$	$\dfrac{1}{n+1}u^{n+1}+C$	$\int \csc^2(u)\,du$	$=$	$-\cot(u)+C$
$\displaystyle\int \frac{1}{u}\,du$	$=$	$\ln\lvert u\rvert+C$	$\int \sec(u)\tan(u)\,du$	$=$	$\sec(u)+C$
$\int e^u\,du$	$=$	e^u+C	$\int \csc(u)\cot(u)\,du$	$=$	$-\csc(u)+C$
$\int b^u\,du$	$=$	$\dfrac{1}{\ln(b)}b^u+C$	$\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du$	$=$	$\sin^{-1}(u)+C$
$\int \sin(u)\,du$	$=$	$-\cos(u)+C$	$\displaystyle\int \frac{1}{1+u^2}\,du$	$=$	$\tan^{-1}(u)+C$
$\int \cos(u)\,du$	$=$	$\sin(u)+C$	$\displaystyle\int \frac{1}{u\sqrt{u^2-1}}\,du$	$=$	$\sec^{-1}\lvert u\rvert+C$

Recall that each of these integration rules is verified by differentiating the answer and checking that the derivative is the same as the integrand.

$$\int f(u)\,du \quad = \quad F(u)+C$$

$$D_u$$

Keep the above list of integration rules in mind, for we will refer to it many times in this chapter.

The above list does not contain the reverse of *every* differentiation rule. For instance, the reverse of the quotient rule $D_u\left[\frac{f(u)}{g(u)}\right] = \frac{f'(u)g(u)-f(u)g'(u)}{\big(g(u)\big)^2}$ is

$$\int \frac{f'(u)g(u)-f(u)g'(u)}{\big(g(u)\big)^2}\,du \;=\; \frac{f(u)}{g(u)}+C.$$

This "rule" is useless (and is not on our list) because we rarely if ever need to find the integral of a function that has the precise form $\frac{f'(u)g(u)-f(u)g'(u)}{\big(g(u)\big)^2}$.

But there is *one* differentiation rule whose reverse is extremely useful. That rule is the chain rule. The reverse of the chain rule is an integration rule called the *substitution rule*. That is this chapter's topic.

44.1 The Substitution Rule

Provided F and g are differentiable, the chain rule (for differentiation) says

$$D_x\Big[F\big(g(x)\big)\Big] \;=\; F'\big(g(x)\big)\cdot g'(x).$$

If $F'(x) = f(x)$, that is, if F is an antiderivative of f, then this becomes

$$D_x\Big[F\big(g(x)\big)\Big] \;=\; f\big(g(x)\big)\cdot g'(x).$$

In other words, if F is an antiderivative of f, then

$$\int f\big(g(x)\big)\cdot g'(x)\,dx \;=\; F\big(g(x)\big)+C. \qquad (*)$$

This is *almost* our new rule, but first we are going to simplify it. Let $\boxed{g(x)=u,}$ so $g'(x)=\frac{du}{dx}$. Multipling both sides of this by the differential dx, $\boxed{g'(x)\,dx = du.}$ Using these boxed equations, replace (or *substitute*) the $g(x)$ in Equation $(*)$ with u, and the $g'(x)\,dx$ with du:

$$\int f\big(g(x)\big)\,g'(x)\,dx = F\big(g(x)\big)+C$$

$$\int f(u)\,du = F(u)+C$$

Because $g(x)=u$, the right-hand sides of these equations are equal. Thus left-hand sides are equal too. That is our new rule.

Substitution Rule If $u = g(x)$, then $\displaystyle\int f\big(g(x)\big)\,g'(x)\,dx = \int f(u)\,du.$

The substitution rule can convert a complicated integral to a simple one.

Substitution Rule If $u = g(x)$, then $\underbrace{\int f\big(g(x)\big)\, g'(x)\, dx}_{\text{complicated}} = \underbrace{\int f(u)\, du}_{\text{simple}}.$

When faced with a complicated integral to which no integration rules apply, the substitution rule—when applicable—lets us make a substitution $u = g(x)$ that converts the integral to a simpler form $\int f(u)\,du$ to which a rule may apply. This is somewhat of an art, and occasionally involves some guesswork and trial and error. (And it definitely requires some practice and experience!) Following are some examples of the substitution rule.

Example 44.1 Find $\displaystyle\int \sin\big(x^2\big)\, 2x\, dx$.

Solution This is does not match any of our familiar integration rules, but the closest match is the rule $\int \sin(u)\,du = -\cos(u) + C$. To make the present problem more like this rule, we can make the substitution $u = x^2$. With this,

$$\int \sin\big(x^2\big)\, 2x\, dx = \int \sin(u)\, 2x\, dx$$

But we still have the $2x\,dx$ where we want just du. (Remember, we are aiming for $\int \sin(u)\,du$.) But also we chose $u = x^2$, and this means $\frac{du}{dx} = 2x$. From this we get $\boxed{du = 2x\,dx.}$ Thus the above $2x\,dx$ actually equals du, and the above turns into the simple integral

$$= \int \sin(u)\, du = -\cos(u) + C.$$

But since $u = x^2$, the u in $-\cos(u) + C$ is actually x^2. We have our answer.

Answer: $\displaystyle\int \sin\big(x^2\big)\, 2x\, dx = -\cos\big(x^2\big) + C.$

To check this answer, notice that $D_x\Big[-\cos\big(x^2\big) + C\Big] = -\sin\big(x^2\big)\,2x.$ ✍

This example illustrates a general approach to using the substitution rule to find an indefinite integral that does not match any known rule: Think of an integration rule that most closely matches the problem. Then make a substitution $u = g(x)$ that moves the problem closer to that rule. Once this choice is made, then $du/dx = g'(x)$, so $du = g'(x)dx$. Look for a $g'(x)dx$ in the problem and replace it with du. If you now have something that matches the rule you aimed for, then substitution has worked.

Example 44.2 Find $\displaystyle\int \frac{\sec\big(\ln(x)\big)\tan\big(\ln(x)\big)}{x}\, dx.$

Solution This integral is similar to the rule $\int \sec(u)\tan(u)\,du = \sec(x)+C$. To make it match this rule, use $\boxed{u=\ln(x)}$ so $\frac{du}{dx}=\frac{1}{x}$, hence $\boxed{du=\frac{1}{x}dx}$. Then

$$\begin{aligned}\int \frac{\sec\big(\ln(x)\big)\tan\big(\ln(x)\big)}{x}\,dx &= \int \sec\big(\ln(x)\big)\tan\big(\ln(x)\big)\frac{1}{x}\,dx\\ &= \int \sec(u)\tan(u)\,du\\ &= \sec(u)+C\\ &= \sec\big(\ln(x)\big)+C\end{aligned}$$

✍

We can check the the answer we just got by differentiating it to see if we get the integrand. By the chain rule,

$$D_x\Big[\sec\big(\ln(x)\big)+C\Big] = \sec\big(\ln(x)\big)\tan\big(\ln(x)\big)\frac{1}{x}+0 = \frac{\sec\big(\ln(x)\big)\tan\big(\ln(x)\big)}{x}.$$

This is indeed the integrand, so our answer is correct. You can check all the examples in this section the same way.

Example 44.3 Find $\displaystyle\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$.

Solution This integral is similar to the rule $\int e^u\,du = e^u+C$. To make it match this rule, use $\boxed{u=\sqrt{x}}$ so $\frac{du}{dx}=\frac{1}{2\sqrt{x}}$, hence $\boxed{du=\frac{1}{2\sqrt{x}}\,dx}$. Then

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx &= \int e^{\sqrt{x}}\frac{1}{2\sqrt{x}}\,dx\\ &= \int e^u\,du = e^u+C = e^{\sqrt{x}}+C.\end{aligned}$$

✍

Example 44.4 Find $\displaystyle\int \sqrt{e^x}\,e^x\,dx$.

Solution This is $\displaystyle\int \big(e^x\big)^{1/2}e^x\,dx$, which resembles $\displaystyle\int u^{1/2}\,du$, to which the power rule applies. So let $\boxed{u=e^x}$. Then $\frac{du}{dx}=e^x$, hence $\boxed{du=e^x\,dx}$. Then

$$\begin{aligned}\int \sqrt{e^x}\,e^x\,dx &= \int \big(e^x\big)^{1/2}e^x\,dx\\ &= \int u^{1/2}\,du = \frac{1}{1/2+1}u^{1/2+1}+C = \frac{2}{3}\sqrt{u}^3+C = \frac{2}{3}\sqrt{e^x}^3+C.\end{aligned}$$

✍

Example 44.5 Find $\displaystyle\int \frac{3x^2+8x-1}{x^3+4x^2-x+1}\,dx$.

Solution Notice that the numerator of the integrand happens to be the derivative of the denominator. Thus suggests $\boxed{u = x^3+4x^2-x+1}$, so $\frac{du}{dx} = 3x^2+8x-1$, and hence $\boxed{du = \left(3x^2+8x-1\right)dx}$. Then

$$\begin{aligned}\int \frac{3x^2+8x-1}{x^3+4x^2-x+1}\,dx &= \int \frac{1}{x^3+4x^2-x+1}\left(3x^2+8x-1\right)dx = \int \frac{1}{u}\,du \\ &= \ln|u| + C = \ln\left|x^3+4x^2-x+1\right| + C.\end{aligned}$$

✍

The substitution rule applies only to integrals that have the exact form $\int f\left(g(x)\right)\cdot g'(x)\,dx$, or those that can be put into this form algebraically. Once the substitution $u = g(x)$ is made, the integral has the simpler form $\int f(u)\,du$. After some practice, when confronted with an integral to which substitution applies you may immediately see the answer without actually doing the substitution. That is good.

But often the structure $\int f\left(g(x)\right)\cdot g'(x)\,dx$ is not immediately apparent, and you may have to do some intelligent guessing to come up with workable substitution $u = g(x)$. Sometimes further algebraic manipulation is needed to attain a familiar form $\int f(u)\,du$. We now work some examples of this type.

Example 44.6 Find $\displaystyle\int \sin\left(x^2\right) x\,dx$.

Solution This looks almost identical to Example 44.1. And as in that example, the rule $\int \sin(u)\,du = -\cos(u) + C$ seems to be the closest match, so try $\boxed{u = x^2}$. From this $\frac{du}{dx} = 2x$, so $\boxed{du = 2x\,dx}$. But here we have a problem. We'd like to replace the $x\,dx$ at the end of $\int \sin\left(x^2\right) x\,dx$ with du, but $du \neq x\,dx$. The $du = 2x\,dx$ that came from our choice $u = x^2$ doesn't match the $x\,dx$ in the integral. To fix the problem, just divide $du = 2x\,dx$ by 2 to get $\frac{1}{2}du = x\,dx$. Now we know the $x\,dx$ actually equals $\frac{1}{2}du$. Making these substitutions,

$$\begin{aligned}\int \sin\left(x^2\right) x\,dx &= \int \sin(u)\frac{1}{2}\,du \\ &= \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\sin\left(x^2\right) + C.\end{aligned}$$

✍

Example 44.7 Find $\displaystyle\int \frac{\sec^2\left(\sqrt{x}\right)\tan^2\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$.

Solution After some thought, you may hit upon this idea: $\boxed{u = \tan\left(\sqrt{x}\right)}$. By the chain rule $\frac{du}{dx} = \sec^2(\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{\sec^2(\sqrt{x})}{2\sqrt{x}}$. Solve this as $\boxed{2\,du = \frac{\sec^2(\sqrt{x})}{\sqrt{x}}dx}$. Now we have

$$\begin{aligned}
\int \frac{\sec^2\left(\sqrt{x}\right)\tan^2\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= \int \left(\tan\left(\sqrt{x}\right)\right)^2 \frac{\sec^2\left(\sqrt{x}\right)}{\sqrt{x}}\,dx \\
&= \int u^2 2\,du \;=\; 2\int u^2\,du \;=\; 2\frac{u^3}{3}+C \\
&= \frac{2\left(\tan\left(\sqrt{x}\right)\right)^3}{3}+C \;=\; \frac{2\tan^3\left(\sqrt{x}\right)}{3}+C
\end{aligned}$$

✍

Example 44.8 Find $\displaystyle\int \cos(x)\sin(x)\,dx$.

Solution Choose $\boxed{u = \cos(x)}$, so $\frac{du}{dx} = -\sin(x)$. Solve this as $\boxed{-du = \sin(x)\,dx}$ to get a match for the $\sin(x)dx$ that appears in the integral. Then

$$\int \cos(x)\sin(x)\,dx \;=\; \int u(-du) \;=\; -\int u\,du \;=\; -\frac{u^2}{2}+C \;=\; -\frac{\cos^2(x)}{2}+C.$$

That is our answer, but here is another approach. Choose $\boxed{u = \sin(x)}$, so $\frac{du}{dx} = \cos(x)$. Solve this as $\boxed{du = \cos(x)\,dx}$. With these substitutions we get

$$\int \cos(x)\sin(x)\,dx \;=\; \int \sin(x)\cos(x)\,dx \;=\; \int u\,du \;=\; \frac{u^2}{2}+C \;=\; \frac{\sin^2(x)}{2}+C.$$

So we got an answer in two different ways. But the answers look different. To see why this is not a mistake, note $\sin^2(x)+\cos^2(x) = 1$, so $\sin^2(x) = 1-\cos^2(x)$. With this, our second answer is

$$\frac{\sin^2(x)}{2}+C \;=\; \frac{1-\cos^2(x)}{2}+C \;=\; -\frac{\cos^2(x)}{2}+\frac{1}{2}+C.$$

So the two answer *are* different: one is just the other plus 1/2. Since we regard the constant as arbitrary, we just write it as C in both cases. ✍

Next we'll do two examples that are very simple; so simple that students often make hasty mistakes when doing integrals of this type. We will use substitution to solve them here, but after some practice you will do this kind of problem mentally, getting the answer in one step.

Example 44.9 Find $\displaystyle\int e^{-x}\,dx$.

Solution This closely resembles the rule $\int e^u\,du = e^u + C$, so to convert to that form we choose $\boxed{u=-x}$. Then $\frac{du}{dx} = -1$, so $\boxed{dx = -du}$. Making these substitutions, $\displaystyle\int e^{-x}\,dx = \int e^u(-du) = -\int e^u\,du = -e^u + C = -e^{-x} + C$. ✍

Example 44.10 Find $\displaystyle\int \cos(2x)\,dx$.

Solution This closely resembles the rule $\int \cos(u)du = \sin(u)+C$, so to convert to that form we choose $\boxed{u=2x}$. Then $\frac{du}{dx} = 2$, so $\boxed{dx = \frac{1}{2}du}$. Then $\displaystyle\int \cos(2x)\,dx = \int \cos(u)\frac{1}{2}\,du = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u)+C$. ✍

Example 44.11 Find $\displaystyle\int x\sqrt{x+1}\,dx$.

Solution Since this looks somewhat like $\int \sqrt{u}\,du$ (which we can do with the power rule for integration), put $\boxed{u=x+1}$. Then $\frac{du}{dx} = 1$, so $\boxed{du = dx}$. Then

$$\int x\sqrt{x+1}\,dx \;=\; \int x(x+1)^{1/2}\,dx \;=\; \int x u^{1/2}\,du.$$

This looks fine, except that there is still an x in the picture, and it didn't go away with the substitution for du. To take care of this, notice because we have made the substitution $u = x+1$, it follows that $x = u-1$. With this our computation continues as

$$\begin{aligned}
\int x u^{1/2}\,du \;=\; \int (u-1)u^{1/2}\,du \;&=\; \int \left((u\cdot u^{1/2} - u^{1/2}\right)\,du \\
&= \int \left(u^{3/2} - u^{1/2}\right)\,du \\
&= \frac{1}{3/2+1}u^{3/2+1} + \frac{1}{1/2+1}u^{1/2+1} + C \\
&= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{5}\sqrt{u}^5 + \frac{2}{3}\sqrt{u}^3 + C \\
&= \frac{2}{5}\sqrt{x+1}^5 + \frac{2}{3}\sqrt{x+1}^3 + C
\end{aligned}$$

✍

It is important to practice some exercises before moving to the next section.

44.2 Substitution in Definite Integrals

Part 2 of the fundamental theorem of calculus states

$$\int_a^b f(x)\,dx \;=\; \Big[F(x)\Big]_a^b,$$

where $F(x) = \int f(x)\,dx$ (with $C = 0$). So finding a *definite* integral involves finding an *indefinite* integral. Sometimes the indefinite integral requires substitution. This section explains the process.

To motivate the topic, we will do our next example two ways. The first way is longer. The second, shorter solution highlights a preferred shortcut.

Example 44.12 Find $\int_0^1 \left(x^2+1\right)^5 2x\,dx$.

Solution A Applying the fundamental theorem of calculus yields

$$\int_0^1 \left(x^2+1\right)^5 2x\,dx \;=\; \Big[F(x)\Big]_0^1 \qquad (*)$$

Before going further, we need to compute $F(x) = \int \left(x^2+1\right)^5 2x\,dx$. For this, let $\boxed{u = x^2+1}$, so $\frac{du}{dx} = 2x$, and $\boxed{du = 2x\,dx}$. Then

$$F(x) \;=\; \int \left(x^2+1\right)^5 2x\,dx \;=\; \int u^5\,du \;=\; \frac{u^6}{6} = \frac{(x^2+1)^6}{6}.$$

We now get an answer by inserting $F(x) = \left(x^2+1\right)^6/6$ into Equation $(*)$, above.

$$\int_0^1 \left(x^2+1\right)^5 2x\,dx \;=\; \left[\frac{(x^2+1)^6}{6}\right]_0^1 \;=\; \frac{(1^2+1)^6}{6} - \frac{(0^2+1)^6}{6} = \frac{63}{6} = \frac{21}{2}.$$

Solution B This time we'll do the substitutions in-line, rather than finding $F(x)$ separately, as above. As before, let $\boxed{u = x^2+1}$, so $\boxed{du = 2x\,dx}$. Then

$$\int_0^1 \left(x^2+1\right)^5 2x\,dx \;=\; \int_{x=0}^{x=1} u^5 du$$

We have almost totally switched from x to u. But the limits of integration are still x-values. (In the original integral, x goes from 0 to 1.) We have emphasized this by writing them as x=0 and x=1. They must be converted to u. Since u=x^2+1, x=0 gives u=0^2+1, and x=1 gives u=1^2+1. Our computation continues:

$$= \int_{u=0^2+1}^{u=1^2+1} u^5 du \;=\; \int_1^2 u^5\,du = \left[\frac{u^6}{6}\right]_1^2 \;=\; \frac{2^6}{6} - \frac{1^6}{6} \;=\; \frac{63}{6} \;=\; \frac{21}{2}.$$

We have now found $\int_0^1 \left(x^2+1\right)^5 2x\,dx$ in two different ways, getting the same answer both times. ✍

The moral of this example is that if we need to do a substitution $u = g(x)$ in a definite integral, then it is quicker to do it in-line, as in Solution B. However, in doing this we must remember to change the limits of integration a and b to $g(a)$ and $g(b)$.

We summarize this discussion by pairing the previous section's substitution rule (for indefinite integrals) with its companion for definite integrals.

Substitution Rule for Indefinite Integrals

If $u = g(x)$, then $\displaystyle\int f\big(g(x)\big)\,g'(x)\,dx = \int f(u)\,du.$

Substitution Rule for Definite Integrals

If $u = g(x)$, then $\displaystyle\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$

Example 44.13 Find $\displaystyle\int_1^2 \frac{5}{(5x-1)^2}\,dx$.

Solution Let $\boxed{u = 5x-1}$, so $\frac{du}{dx} = 5$ and $\boxed{du = 5\,dx}$. Then

$$\int_1^2 \frac{5}{(5x-1)^2}\,dx = \int_1^2 (5x-1)^{-2}\,5\,dx = \int_{5\cdot1-1}^{5\cdot2-1} u^{-2}\,du = \int_4^9 u^{-2}\,du = \left[\frac{-1}{u}\right]_4^9 = \frac{5}{36}.$$ ✍

Example 44.14 Find $\displaystyle\int_{-1}^0 \frac{7}{x^2+2x+2}\,dx$.

Solution Perhaps your first impulse is to put $u = x^2+2x+2$, but then $du = (2x+1)\,dx$ is nowhere in sight. After some trial and error you may notice that this integral is similar to $\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C$. In fact,

$$\int_{-1}^0 \frac{7}{x^2+2x+2}\,dx = 7\int_{-1}^0 \frac{1}{1+(x^2+2x+1)}\,dx = 7\int_{-1}^0 \frac{1}{1+(x+1)^2}\,dx$$

So let $\boxed{u = x+1}$. Then $\frac{du}{dx} = 1$, so $\boxed{du = dx}$. The computation continues as:

$$= 7\int_{-1+1}^{0+1} \frac{1}{1+u^2}\,du = 7\Big[\tan^{-1}(u)\Big]_0^1 = 7\left(\frac{\pi}{4} - 0\right) = \frac{7\pi}{4}.$$ ✍

Example 44.15 Find $\displaystyle\int_0^1 \frac{x+1}{x^2+2x+2}\,dx$.

Solution Let $\boxed{u = x^2+2x+2}$. Then $\frac{du}{dx} = 2x+2$, so $\boxed{\frac{1}{2}du = (x+1)\,dx}$. Then

$$\begin{aligned}
\int_0^1 \frac{x+1}{x^2+2x+2}\,dx &= \int_0^1 \frac{1}{x^2+2x+2}(x+1)\,dx \\
&= \int_{0^2+2\cdot 0+2}^{1^2+2\cdot 1+2} \frac{1}{u}\frac{1}{2}\,du \;=\; \frac{1}{2}\int_2^5 \frac{1}{u}\,du \;=\; \frac{1}{2}\Big[\ln|u|\Big]_2^5 \\
&= \frac{1}{2}\big(\ln(5)-\ln(2)\big) \;=\; \frac{1}{2}\ln\left(\frac{5}{2}\right) \;=\; \ln\sqrt{\frac{5}{2}}.
\end{aligned}$$

✍

Exercises for Chapter 44

Find the integrals.

1. $\displaystyle\int 6\theta\cos(3\theta^2)\,d\theta$

2. $\displaystyle\int e^{2x^2}4x\,dx$

3. $\displaystyle\int \frac{4x}{2x^2-6}\,dx$

4. $\displaystyle\int 12s^2\sqrt{4s^3+15}\,ds$

5. $\displaystyle\int \frac{3x^2+2x+1}{x^3+x^2+x}\,dx$

6. $\displaystyle\int \frac{4x}{e^{2x^2}}\,dx$

7. $\displaystyle\int \frac{\sec^2(-1/x)}{x^2}\,dx$

8. $\displaystyle\int \frac{1}{\sqrt{x}\left(1+2\sqrt{x}\right)^3}\,dx$

9. $\displaystyle\int \sin^6(x)\cos(x)\,dx$

10. $\displaystyle\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

11. $\displaystyle\int 2e^{-x}\,dx$

12. $\displaystyle\int 4\sin(3x)\,dx$

13. $\displaystyle\int \frac{\sin(2x)}{\cos^5(2x)}\,dx$

14. $\displaystyle\int \sin(x)e^{\cos(x)}\,dx$

15. $\displaystyle\int x\sqrt{1-x^2}\,dx$

16. $x^2\cos\left(x^3\right)\,dx$

17. $\displaystyle\int 2\sin(2x)\big(1-\cos(2x)\big)\,dx$

18. $\displaystyle\int \frac{6x^2+6}{\left(x^4+4x\right)}\,dx$

19. $\displaystyle\int \pi\sin^2(\pi x)\cos(\pi x)\,dx$

20. $\displaystyle\int \frac{\cos(6x)}{\sqrt{\sin(6x)}}\,dx$

21. $\displaystyle\int \frac{\cos\left(1/x^2\right)}{x^3}\,dx$

22. $\displaystyle\int \frac{2x^9 - e^x}{x^{10} - 5e^x}\,dx$

23. $\displaystyle\int_1^3 \frac{3x^2 + 2x + 1}{x^3 + x^2 + x}\,dx$

24. $\displaystyle\int_0^1 \left(x^4 + 1\right)\sqrt{x^5 - 5x + 4}\,dx$

25. $\displaystyle\int_0^3 e^{-5x}\,dx$

26. $\displaystyle\int_1^2 \frac{x+1}{\left(x^2 + 2x\right)^2}\,dx$

27. $\displaystyle\int_0^1 x(x^2+1)^5\,dx$

28. $\displaystyle\int_{-\pi/6}^{\pi/6} \tan(2x)\sec(2x)\,dx$

29. $\displaystyle\int_2^1 \cos\left(\frac{\pi x}{2}\right)dx$

30. $\displaystyle\int_0^1 x\sqrt{x^2+1}\,dx$

31. $\displaystyle\int_{-1}^0 \frac{y}{1+y^2}\,dy$

32. $\displaystyle\int_0^3 \frac{dx}{x+1}$

33. $\displaystyle\int_0^{\pi/2} \sin^4(3x)\cos(3x)\,dx$

34. $\displaystyle\int_0^1 \frac{5}{\left(5x+1\right)^2}\,dx$

35. $\displaystyle\int_0^{\sqrt{\pi/4}} \sec^2\left(x^2\right)x\,dx$

36. $\displaystyle\int_{-2}^1 \frac{3}{3x+7}\,dx$

37. Find the area under the graph of $\sec^2(2x)$ between 0 and $\pi/8$.

38. Find the area under the graph of $x\sin(x^2)$ between 0 and $\sqrt{\pi/6}$.

39. Find $\int \sin(x)\cos(x)\,dx$ using the substitution $u = \sin(x)$. Then find $\int \sin(x)\cos(x)\,dx$ using the substitution $u = \cos(x)$. Explain why it is not a contradiction that the answers look different.

Exercise Solutions for Chapter 44

1. Let $u = 3\theta^2$, so $\frac{du}{d\theta} = 6\theta$ and $du = 6\theta\, d\theta$. Then

$$\int 6\theta\cos(3\theta^2)\,d\theta = \int \cos(3\theta^2)6\theta\,d\theta = \int \cos(u)\,du = \sin(u)+C = \sin(3\theta^2)+C$$

3. Let $u = 2x^2-6$ so $\frac{du}{dx} = 4x$ and $du = 4x\,dx$. Then

$$\int \frac{4x}{2x^2-6}\,dx = \int \frac{1}{2x^2-6}4x\,dx = \int \frac{1}{u}\,du = \ln|u|+C = \ln|2x^2-6|+C$$

5. Let $u = x^3+x^2+x$, so $\frac{du}{dx} = 3x^2+2x+1$, hence $du = \left(3x^2+2x+1\right)dx$. Then

$$\int \frac{3x^2+2x+1}{x^3+x^2+x}\,dx = \int \frac{1}{x^3+x^2+x}\left(3x^2+2x+1\right)dx = \int \frac{1}{u}\,du = \ln|u|+C = \ln|x^3+x^2+x|+C$$

7. Let $u = -1/x$ so $\frac{du}{dx} = \frac{1}{x^2}$, and $du = \frac{1}{x^2}dx$. Then

$$\int \frac{\sec^2(-1/x)}{x^2}\,dx = \int \sec^2(-1/x)\frac{1}{x^2}\,dx = \int \sec^2(u)\,du = \tan(u)+C = \tan(-1/x)+C$$

9. Let $u = \sin(x)$ so $\frac{du}{dx} = \cos(x)$ and $du = \cos(x)\,dx$ Then

$$\int \sin^6(x)\cos(x)\,dx = \int u^6\,du = \frac{u^7}{7}+C = \frac{\sin^7(x)}{7}+C$$

11. Let $u = -x$ so $\frac{du}{dx} = -1$ and $-du = dx$.
Then $\int 2e^{-x}\,dx = 2\int e^{-x}\,dx = 2\int e^{u}(-du)2 = -2\int e^{u}\,du = -2e^{u}+C = -2e^{-x}+C$

13. Let $u = \cos(2x)$ so $\frac{du}{dx} = -2\sin(2x)$ and $-\frac{1}{2}\,du = \sin(2x)\,dx$. Then

$$\int \frac{\sin(2x)}{\cos^5(2x)}\,dx = \int (\cos(2x))^{-5}\sin(2x)\,dx = \int u^{-5}\left(-\frac{1}{2}du\right) = \frac{1}{2}\int u^{-5}\,du = \frac{1}{2}\cdot\frac{u^{-4}}{-4}+C$$

$$-\frac{1}{8u^4}+C = -\frac{1}{8\cos^4(2x)}+C$$

15. Let $u = 1-x^2$, so $\frac{du}{dx} = -2x$, hence $-\frac{1}{2}du = x\,dx$. Then

$$\int x\sqrt{1-x^2}\,dx = \int \left(1-x^2\right)^{1/2}x\,dx = \int u^{1/2}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\frac{2}{3}u^{3/2}+C$$

$$-\frac{\sqrt{u}^3}{3}+C \;=\; -\frac{\sqrt{1-x^2}^3}{3}+C$$

17. Let $u = 1-\cos(2x)$, so $du = 2\sin(2x)\,dx$. Then $\displaystyle\int 2\sin(2x)\big(1-\cos(2x)\big)\,dx =$

$$\int \big(1-\cos(2x)\big)2\sin(2x)\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{\big(1-\cos(2x)\big)^2}{2}+C$$

19. Let $u = \sin(\pi x)$ then $du = \cos(\pi x)\pi\,dx$. Then $\displaystyle\int \pi\sin^2(\pi x)\cos(\pi x)\,dx =$

$$\int \sin^2(\pi x)\pi\cos(\pi x)\,dx = \int u^2\,du = \frac{u^3}{3}+C = \frac{\sin^3(\pi x)}{3}+C$$

21. Let $u = 1/x^2$, so $du = -\frac{2}{x^3}\,dx$ and so $-\frac{1}{2}du = \frac{1}{x^3}\,dx$. Then $\int \frac{\cos\left(1/x^2\right)}{x^3}\,dx =$

$$\int \cos\left(1/x^2\right)\frac{1}{x^3}\,dx = \int \cos(u)\frac{-1}{2}\,du = -\frac{1}{2}\int \cos(u)\,du = -\frac{1}{2}\sin(u)+C = -\frac{1}{2}\sin\left(\frac{1}{x^2}\right)+C$$

23. Let $u = x^3+x^2+x$, so $\frac{du}{dx} = 3x^2+2x+1$, hence $du = \left(3x^2+2x+1\right)dx$. Then

$$\int_1^3 \frac{3x^2+2x+1}{x^3+x^2+x}\,dx = \int_1^3 \frac{1}{x^3+x^2+x}\left(3x^2+2x+1\right)dx = \int_{1^3+1^2+1}^{3^3+3^2+3}\frac{1}{u}du = \int_3^{39}\frac{1}{u}du$$

$$= \Big[\ln|u|\Big]_3^{39} = \ln|39|-\ln|3| = \ln\left(\frac{39}{3}\right) = \ln(13)$$

25. Let $u = -5x$ so $\frac{du}{dx} = -5$ and $-\frac{1}{5}du = dx$. Then

$$\int_0^3 e^{-5x}\,dx = \int_{-5\cdot 0}^{-5\cdot 3} e^u\left(-\frac{1}{5}du\right) = -\frac{1}{5}\int_0^{-15} e^u\,du = -\frac{1}{5}\Big[e^u\Big]_0^{-15} = -\frac{1}{5}\left(e^{-15}-e^0\right) = \frac{1}{5}-\frac{1}{5e^{15}}$$

27. Let $u = x^2+1$, so $\frac{du}{dx} = 2x$ and $\frac{1}{2}du = x\,dx$. Then

$$\int_0^1 x(x^2+1)^5\,dx = \int_0^1 (x^2+1)^5 x\,dx = \int_{0^2+1}^{1^2+1} u^5\frac{1}{2}du = \frac{1}{2}\int_1^2 u^5 du = \frac{1}{2}\left[\frac{u^6}{6}\right]_1^2 = \frac{1}{2}\left(\frac{2^6}{6}-\frac{1^6}{6}\right) = \frac{21}{4}$$

29. $$\int_2^1 \cos\left(\frac{\pi x}{2}\right)dx = \left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_2^1 = \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right)-\frac{2}{\pi}\sin(\pi) = \frac{2}{\pi}\cdot 1-\frac{2}{\pi}\cdot 0 = \frac{2}{\pi}$$

31. Let $u = 1+y^2$. Then $du = 2y\,dy$, so $\frac{1}{2}du = y\,dy$ and $\int_{-1}^0 \frac{y}{1+y^2}\,dy = \int_{-1}^0 \frac{1}{1+y^2}y\,dy$

$$= \int_{1+(-1)^2}^{1+0^2}\frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int_2^1 \frac{1}{u}du = \frac{1}{2}\Big[\ln|u|\Big]_1^2 = \frac{1}{2}\Big(\ln|2|-\ln|1|\Big) = \frac{1}{2}\ln(2)$$

33. Let $u = \sin(3x)$. Then $du = \cos(3x)3\,dx$, so $\frac{1}{3}du = \cos(3x)\,dx$ and

$$\int_0^{\pi/2}\sin^4(3x)\cos(3x)\,dx = \int_{\sin(3\cdot 0)}^{\sin(3\cdot\pi/2)} u^4\frac{1}{3}\,du = \frac{1}{3}\int_0^{-1}u^4\,du = \frac{1}{3}\left[\frac{u^5}{5}\right]_0^{-1} = -\frac{1}{15}$$

35. Let $u = x^2$. Then $du = 2x\,dx$, so $\frac{1}{2}du = x\,dx$ and $\int_0^{\sqrt{\pi/4}}\sec^2\left(x^2\right)x\,dx =$

$$\int_{0^2}^{\sqrt{\pi/4}^2}\sec^2(u)\frac{1}{2}\,du = \frac{1}{2}\int_0^{\pi/4}\sec^2(u)\,du = \frac{1}{2}\Big[\tan(u)\Big]_0^{\pi/4} = \frac{1}{2}\Big(\tan(\pi/4)-\tan(0)\Big) = \frac{1}{2}.$$

37. Find the area under the graph of $\sec^2(2x)$ between 0 and $\pi/8$.

The answer will be $\int_0^{\pi/8}\sec^2(2x)\,dx$. Let $u = 2x$, so $du = 2\,dx$ and $\frac{1}{2}du = dx$. Then

$$\int_0^{\pi/8}\sec^2(2x)\,dx = \int_{2\cdot 0}^{2\cdot\pi/8}\sec^2(u)\frac{1}{2}\,du = \frac{1}{2}\int_0^{\pi/4}\sec^2(u)\,du = \frac{1}{2}\Big(\tan(\pi/4)-\tan(0)\Big) = \frac{1}{2}.$$

39. If $u = \sin(x)$, then $du = \cos(x)\,dx$, so $\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{\sin^2(x)}{2}+C.$

If $u = \cos(x)$, then $-du = \sin(x)\,dx$, so $\int \sin(x)\cos(x)\,dx = -\int u\,du = -\frac{u^2}{2}+C = -\frac{\cos^2(x)}{2}+C.$

These two answers look different, but this is not a contradiction. Add a constant of 1/2 to the second answer and you get $\frac{1}{2} - \frac{\cos^2(x)}{2} + C = \frac{1-\cos^2(x)}{2} + C = \frac{\sin^2(x)}{2} + C$. So the first answer is just 1/2 plus the second answer, and the 1/2 gets absorbed into the C.