We know that \( \frac{d}{dx}[e^x] = e^x \). But what about derivatives of exponential function with bases other than \( e \)? In other words, what is \( \frac{d}{dx}[a^x] \)? And what about \( \frac{d}{dx}[\ln(x)] \) and \( \frac{d}{dx}[\log_a(x)] \)? The main goal of this chapter is to answer these questions and thus expand our list of derivative rules.

Let’s start with \( \frac{d}{dx}[a^x] \). Since \( \ln(x) \) is the inverse of \( e^x \), we know \( a = e^{\ln(a)} \). We can thus convert the power \( a^x \) to a power of \( e \):

\[
a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a)x}.
\]

With this, we can get the derivative of \( a^x \) with the chain rule:

\[
\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{\ln(a)x}] = e^{\ln(a)x} \cdot \frac{d}{dx}[\ln(a)x] = e^{\ln(a)x} \ln(a) = a^x \ln(a).
\]

So the derivative of \( a^x \) is just \( a^x \) times the constant \( \ln(a) \). This is a new rule.

**Rule 16** \( \frac{d}{dx}[a^x] = \ln(a)a^x \)

For example, \( \frac{d}{dx}[10^x] = \ln(10)10^x \approx 2.302 \cdot 10^x \). Also \( \frac{d}{dx}[2^x] = \ln(2)2^x \approx 0.693 \cdot 2^x \). Notice how special the base \( e \) is: \( \frac{d}{dx}[e^x] = \ln(e)e^x = 1 \cdot e^x = e^x \).

The base \( a = e \) is the only base for which the derivative of \( a^x \) is 1 times \( a^x \).

Next we will get a rule for \( \frac{d}{dx}[\ln(x)] \). Our strategy will be to use the fact that \( \ln(x) \) is the inverse of \( e^x \), that is,

\[
\text{if } f(x) = e^x, \text{ then } f^{-1}(x) = \ln(x).
\]

Our plan is to first develop a general rule for \( \frac{d}{dx}[f^{-1}(x)] \) and then use it to get \( \frac{d}{dx}[\ln(x)] \). (See Chapter 4 If you need to review inverses.)
Thus our immediate question is: What is \( \frac{d}{dx} f^{-1}(x) \)?

To answer this, think about the relation between \( f \) and its inverse \( f^{-1} \):

\[
f(f^{-1}(x)) = x.
\]

The two sides of this equation are equal functions, so if we differentiate both sides the derivatives will be equal:

\[
\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x
\]

The right-hand side of this equation is 1. The left-hand side is the derivative of a composition, so we can apply the chain rule to it:

\[
f'(f^{-1}(x)) \frac{d}{dx} f^{-1}(x) = 1
\]

In applying the chain rule we multiplied \( f'(f^{-1}(x)) \) by \( \frac{d}{dx} [f^{-1}(x)] \). We stopped there because we don’t know what \( \frac{d}{dx} [f^{-1}(x)] \) is. But it’s exactly what we want to find! We can isolate it by dividing the above equation by \( f'(f^{-1}(x)) \):

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.
\]

This is our latest rule.

**Rule 17 (The inverse rule)** If \( f \) is a function having a derivative \( f' \) and an inverse \( f^{-1} \), then

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.
\]

To illustrate this rule, suppose \( f(x) = x^3 \), which has an inverse \( f^{-1}(x) = \sqrt[3]{x} \).

Let’s find \( \frac{d}{dx} [f^{-1}(x)] \). We know that \( f'(x) = 3x^2 \), so our new rule gives

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3(f^{-1}(x))^2} = \frac{1}{3\sqrt[3]{x^2}}
\]

Granted, this is not all that impressive, since we can use the power rule to get the same answer:

\[
\frac{d}{dx} f^{-1}(x) = \frac{d}{dx} [\sqrt[3]{x}] = \frac{d}{dx} \left[ x^{\frac{1}{3}} \right] = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}.
\]
But the inverse rule can be very useful. We’ll now use it to find the derivative of \( \ln(x) \). Say \( f(x) = e^x \), so \( f^{-1}(x) = \ln(x) \). Then

\[
\frac{d}{dx} \left[ \ln(x) \right] = \frac{d}{dx} \left[ f^{-1}(x) \right]
\]

because \( \ln(x) = f^{-1}(x) \)

\[
= \frac{1}{f'(f^{-1}(x))}
\]

by inverse rule

\[
= \frac{1}{f'(\ln(x))}
\]

because \( f^{-1}(x) = \ln(x) \)

\[
= \frac{1}{e^{\ln(x)}}
\]

because \( f'(x) = e^x \)

\[
= \frac{1}{x}
\]

because \( e^{\ln(x)} = x \).

Thus \( \frac{d}{dx} \left[ \ln(x) \right] = \frac{1}{x} \). Figure 24.1 (left) illustrates this remarkable fact. It shows the function \( f(x) = \ln(x) \) along with its derivative \( f'(x) = \frac{1}{x} \). Notice how if \( x \) is near 0, the tangent to \( \ln(x) \) at \( x \) is very steep, and indeed the derivative \( \frac{1}{x} \) is very large. But as \( x \) gets bigger, the tangent to \( \ln(x) \) gets closer to horizontal (slope 0) while the derivative \( \frac{1}{x} \) approaches zero.

Figure 24.1. Left: the graphs of \( f(x) = \ln(x) \) (black) and \( f'(x) = \frac{1}{x} \) (blue) with domain \((0, \infty)\). Right: the graphs of \( f(x) = \ln|x| \) (black) and \( f'(x) = \frac{1}{x} \) (blue).

Notice however, that the domain of \( \ln(x) \) is \((0, \infty)\) but the domain of \( \frac{1}{x} \) is \((-\infty, 0) \cup (0, \infty)\). So when we say that the derivative of \( \ln(x) \) is \( \frac{1}{x} \), we really mean \( \frac{1}{x} \) with its domain restricted to \((0, \infty)\). Figure 24.1 (right) shows a somewhat more complete scenario. It shows the function \( \ln(|x|) \), which we will abbreviate as \( \ln|x| \). This function has domain \((-\infty, 0) \cup (0, \infty)\), and its derivative is \( \frac{1}{x} \) with its usual domain. So our latest rule has two parts.

**Rule 18** \( \frac{d}{dx} \left[ \ln(x) \right] = \frac{1}{x} \) and \( \frac{d}{dx} \left[ \ln|x| \right] = \frac{1}{x} \).
Here it is understood that in the first formula the domain of $\ln(x)$ and $\frac{1}{x}$ is $(0, \infty)$. In the second formula the domain of $\ln |x|$ and $\frac{1}{x}$ is all real numbers except 0. Do not sweat the difference between the two versions of this rule – they say almost the same thing, and the second implies the first. We will mostly use the first version in parts 3 and 4 of this book, but the second version becomes particularly useful in Part 5.

At the beginning of this chapter we said our main goals were to find formulas for $\frac{d}{dx}[a^x]$, $\frac{d}{dx}[\ln(x)]$ and $\frac{d}{dx} [\log_a(x)]$. We’ve done all but the last one. For it we will use the change of base formula (Fact 5.1 in Chapter 5, page 88) which states

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$  

Using this, the constant multiple rule and Rule 18, we get

$$\frac{d}{dx} [\log_a(x)] = \frac{1}{x \ln(a)} \cdot \frac{d}{dx} [\ln(x)] = \frac{1}{x \ln(a)}.$$  

With our prior agreement about domains, we get another two-part formula.

**Rule 19**  

$$\frac{d}{dx} [\log_a(x)] = \frac{1}{x \ln(a)} \quad \text{and} \quad \frac{d}{dx} [\log_a |x|] = \frac{1}{x \ln(a)}.$$  

**Example 24.1**  

$$\frac{d}{dx} [\log_3(x) \tan(x)] = \frac{d}{dx} [\log_3(x)] \tan(x) + \log_3(x) \frac{d}{dx} [\tan(x)]$$  

(product rule)  

$$= \frac{1}{x \ln(3)} \tan(x) + \log_3(x) \sec^2(x).$$  

**Example 24.2**  

Find $\frac{d}{dx} [\sqrt{5 + x^3 + \ln(x)}]$.  

This is the derivative of a function to a power, so we can use the generalized power rule:

$$\frac{d}{dx} \left[ a^{\ln(f(x))} \right] = a^{\ln(f(x))} \frac{d}{dx} \left[ \ln(f(x)) \right].$$  

$$\frac{d}{dx} [\sqrt{5 + x^3 + \ln(x)}] = \frac{1}{2} (5 + x^3 + \ln(x))^{-1/2} \frac{d}{dx} [5 + x^3 + \ln(x)]$$  

$$= \frac{1}{2} (5 + x^3 + \ln(x))^{-1/2} \left[ 3x^2 + \frac{1}{x} \right]$$  

$$= \frac{3x^2 + \frac{1}{x}}{2\sqrt{5 + x^3 + \ln(x)}}.$$  

$$\Phi$$
Example 24.3 \( \frac{d}{dx} \left[ 7 + x + (\ln(x))^3 \right] = 0 + 1 + 3(\ln(x))^2 \frac{d}{dx} \ln(x) \)
\[= 1 + 3(\ln(x))^2 \frac{1}{x} = \frac{1 + 3(\ln(x))^2}{x}. \]

Example 24.4 \( \frac{d}{dx} \left[ \frac{\ln(x)}{x} \right] = \frac{\frac{d}{dx} \ln(x) \cdot x + \ln(x) \cdot \frac{d}{dx} x}{x^2} = \frac{\frac{1}{x} \cdot x + \ln(x) \cdot 1}{x^2} \)
\[= \frac{1 + \ln(x)}{x^2}. \]

(quotient rule)

Example 24.5 Find the derivative of \(10^{2x^2 + 3x^2 + 2}.\)

The composition \(y = 10^{2x^2 + 3x^2 + 2}\) can be broken up as \(\begin{cases} y = 10^u \\ u = x^2 + 3x + 2. \end{cases}\)

The chain rule then gives \(\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\)
\[= \ln(10) 10^u \cdot (2x + 3 + 0) \]
\[= \ln(10) 10^{x^2 + 3x^2 + 2}(2x + 3). \]

In Example 24.5 we differentiated a function of form \(a^{g(x)}.\) Let’s repeat our steps to get a chain rule generalization for the rule \(\frac{d}{dx} \left[a^{g(x)}\right] = \ln(a) a^{g(x)} g'(x).\)

Example 24.6 Find the derivative of \(a^{g(x)}.\)

The composition \(y = a^{g(x)}\) can be broken up as \(\begin{cases} y = a^u \\ u = g(x). \end{cases}\)

The chain rule then gives \(\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}\)
\[= \ln(a) a^u \cdot g'(x) \]
\[= \ln(a) a^{g(x)} g'(x). \]

This examples shws \(\frac{d}{dx} \left[a^{g(x)}\right] = \ln(a) a^{g(x)} g'(x),\) a companion to the rule \(\frac{d}{dx} \left[e^{g(x)}\right] = e^{g(x)} g'(x).\) We will summarize these and chain rule generalizations of the other rules from this chapter on the bottom of the next page.
Example 24.7  Find the derivative of \( y = \ln|\sin(x)| \).

This is a composition, and the function can be broken up as
\[
\begin{aligned}
 y &= \ln|u| \\
 u &= \sin(x)
\end{aligned}
\]

The chain rule gives
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \cos(x) = \frac{1}{\sin(x)} \cos(x) = \frac{\cos(x)}{\sin(x)}.
\]

Example 24.7 illustrates a common pattern, which is to differentiate a function of from \( \ln|g(x)| \) or \( \ln(g(x)) \). Let’s redo the example in this setting.

Example 24.8  Find the derivative of \( y = \ln|g(x)| \).

This is a composition, and the function can be broken up as
\[
\begin{aligned}
 y &= \ln|u| \\
 u &= g(x)
\end{aligned}
\]

The chain rule gives
\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} g'(x) = \frac{1}{g(x)} g'(x) = \frac{g'(x)}{g(x)}.
\]

Example 24.8 has shown that
\[
\frac{d}{dx} \ln|g(x)| = \frac{1}{g(x)} \cdot \frac{d}{dx} g(x) = \frac{g'(x)}{g(x)}.
\]

This is the chain rule generalization of the rule \( \frac{d}{dx} \ln|x| = \frac{1}{x} \), and it is worth remembering. It implies \( \frac{d}{dx} \ln(g(x)) = \frac{g'(x)}{g(x)} \), and we often use it this way. (Recall \( \ln(g(x)) \) is not defined when \( g(x) \) is negative, so the rule as stated for \( \ln|g(x)| \) is more all-encompassing.)

Here is a summary of this chapter’s main rules, along side their chain rule generalizations. Remember them and internalize them.

**Differentiation rules for exponential and log functions**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Chain rule generalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx} e^x ) = e^x</td>
<td>( \frac{d}{dx} e^{g(x)} ) = e^{g(x)} g'(x)</td>
</tr>
<tr>
<td>( \frac{d}{dx} a^x ) = \ln(a) a^x</td>
<td>( \frac{d}{dx} a^{g(x)} ) = \ln(a) a^{g(x)} g'(x)</td>
</tr>
<tr>
<td>( \frac{d}{dx} \ln</td>
<td>x</td>
</tr>
<tr>
<td>( \frac{d}{dx} \log_a</td>
<td>x</td>
</tr>
</tbody>
</table>

We prefer the base \( e \), so you should expect to that the formulas for \( a^x \) and \( \log_a \) to play less of a role. (Though in computer science, \( \log_2 \) is significant!)
Example 24.9  Find the derivative of \( y = \ln |4x^5 + 6x^3 + x + 3| \).

We will do this in two different ways. First we will use the chain rule, and then we will use the formula \( \frac{d}{dx} \ln |g(x)| = \frac{g'(x)}{g(x)} \) from the previous page.

Using the chain rule, we first break the function up as \[
\begin{align*}
y &= \ln |u| \\
u &= 4x^5 + 6x^3 + x + 3
\end{align*}
\]

The chain rule gives \[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} (20x^4 + 18x^2 + 1)
\]

Next, using the formula \( \frac{d}{dx} \ln |g(x)| = \frac{g'(x)}{g(x)} \), from the previous page, the answer comes in one step:

\[
\frac{d}{dx} \ln |4x^5 + 6x^3 + x + 3| = \frac{20x^4 + 18x^2 + 1}{4x^5 + 6x^3 + x + 3}
\]

Example 24.10  Find the derivative of \( y = \ln |\tan (\sqrt{x^2 + 3x})| \).

This has the form of a composition \( \ln |g(x)| \), so we can use either the straight chain rule or the formula \( \frac{d}{dx} \ln |g(x)| = \frac{g'(x)}{g(x)} \) from the previous page. Let’s try the formula.

\[
\frac{d}{dx} \ln |\tan (\sqrt{x^2 + 3x})| = \frac{1}{\tan (\sqrt{x^2 + 3x})} \frac{d}{dx} \tan (\sqrt{x^2 + 3x})
\]

\[
= \frac{1}{\tan (\sqrt{x^2 + 3x})} \sec^2 (\sqrt{x^2 + 3x}) \frac{d}{dx} \sqrt{x^2 + 3x}
\]

\[
= \frac{1}{\tan (\sqrt{x^2 + 3x})} \sec^2 (\sqrt{x^2 + 3x}) \frac{1}{2} (x^2 + 3x)^{-\frac{1}{2}} \frac{d}{dx} (x^2 + 3x)
\]

\[
= \frac{1}{\tan (\sqrt{x^2 + 3x})} \sec^2 (\sqrt{x^2 + 3x}) \frac{1}{2} (x^2 + 3x)^{-\frac{1}{2}} (2x + 3)
\]

\[
= \frac{(2x + 3) \sec^2 (\sqrt{x^2 + 3x})}{2 \tan (\sqrt{x^2 + 3x}) \sqrt{x^2 + 3x}}
\]
Derivatives of Inverse Functions and Logarithms

Exercises for Chapter 24

In exercises 1–20 differentiate the given function.

1. \( \ln(x) + \frac{1}{x} + \sqrt{x} + 3 \)
2. \( \ln\left( \frac{x^2 + 1}{x} \right) \)
3. \( \frac{\ln(w)}{w} \)
4. \( \frac{1}{x^2 + \ln(x)} \)
5. \( \ln(\sin^3(x)) \)
6. \( \ln(\tan(x)) \)
7. \( 5 + \ln(\pi\theta) + \sqrt{\theta^3} \)
8. \( \ln(\sec(x^3)) \)
9. \( \cos(\ln(x)) \)
10. \( (\sec(\ln(x)))^3 \)
11. \( \pi^2 + \ln(5\theta) + \sqrt{\theta^9} \)
12. \( \ln(\sec(x^3)) \)
13. \( \ln(x^2 + 1)\sqrt{3x + 1} \)
14. \( \sec(\ln(x^3)) \)
15. \( \ln(xe^x) \)
16. \( \ln(x)e^x \)
17. \( \tan(\ln(x)) + x \)
18. \( \ln(\sin(x^3)) \)
19. \( \ln\left( \frac{1}{x} \right) \)
20. \( \frac{x^3 \ln(x)}{x^3 + 1} \)

21. Find \( \lim_{h \to 0} \frac{\ln(3 + h) - \ln(3)}{h} \)
22. Find \( \lim_{z \to 3} \frac{2^z - 8}{z - 3} \)

Exercise Solutions for Chapter 24

In exercises 1–20 differentiate the given function.

1. \( \frac{d}{dx} \left[ \ln(x) + \frac{1}{x} + \sqrt{x} + 3 \right] = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{2\sqrt{x}} \)

3. \( \frac{d}{dw} \left[ \frac{\ln(w)}{w} \right] = \frac{\frac{\ln(w) - 1}{w^2}}{w} = \frac{1 - \ln(w)}{w^2} \)

5. \( \frac{d}{dx} \left[ \ln(\sin^3(x)) \right] = \frac{1}{\sin^3(x)} \frac{d}{dx} \left[ \sin^3(x) \right] = \frac{1}{\sin^3(x)} \cdot 3 \sin^2(x) \cdot \cos(x) = \frac{3 \cos(x)}{\sin(x)} \)

7. \( \frac{d}{d\theta} \left[ 5 + \ln(\pi\theta) + \sqrt{\theta^3} \right] = \frac{d}{d\theta} \left[ 5 + \frac{\ln(\pi\theta) + \theta^{3/2}}{\theta} \right] = \frac{\pi}{\theta} + \frac{3}{2} \theta^{1/2} = \frac{1}{\theta} + \frac{3}{2} \sqrt{\theta} \)

9. \( \frac{d}{dx} \left[ \cos(\ln(x)) \right] = -\sin(\ln(x)) \cdot \frac{1}{x} = -\frac{\sin(\ln(x))}{x} \)

11. \( \frac{d}{dx} \left[ \pi^2 + \ln(5\theta) + \sqrt{\theta^9} \right] = \frac{d}{dx} \left[ \pi^2 + \ln(5\theta) + \theta^{9/2} \right] = 0 + \frac{5}{5\theta} + \frac{9}{2} \theta^{7/2} = \frac{1}{\theta} + \frac{9}{2} \sqrt{\theta} \)
13. \[ \frac{d}{dx} \left[ \ln(x^2 + 1) \sqrt{3x + 1} \right] = \frac{d}{dx} \left[ \ln(x^2 + 1) \right] \sqrt{3x + 1} + \ln(x^2 + 1) \frac{d}{dx} \left[ \sqrt{3x + 1} \right] \]
\[= \frac{2x}{x^2 + 1} \sqrt{3x + 1} + \ln(x^2 + 1) \frac{3}{2\sqrt{3x + 1}} \]

15. \[ \frac{d}{dx} \left[ \ln(xe^x) \right] = \frac{1}{xe^x} \frac{d}{dx} [xe^x] = \frac{1}{xe^x} \left( 1 \cdot e^x + xe^x \right) = \frac{e^x(1 + x)}{xe^x} = \frac{1 + x}{x} \]

17. \[ \frac{d}{dx} \left[ \tan(\ln(x)) + x \right] = \sec^2(\ln(x)) \frac{1}{x} + 1 = \frac{\sec^2(\ln(x))}{x} + 1 \]

19. \[ \frac{d}{dx} \left[ \ln \left( 1 + \frac{1}{x} \right) \right] = \frac{1}{1 + \frac{1}{x}} \frac{d}{dx} \left[ 1 + \frac{1}{x} \right] = \frac{1}{1 + \frac{1}{x}} \left( -\frac{1}{x^2} \right) = -\frac{1}{x^2 + x} \]

21. Find \[ \lim_{h \to 0} \frac{\ln(3 + h) - \ln(3)}{h}. \] Solution: Let \( f(x) = \ln(x). \)

Then \( \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h} = f'(x) = \frac{1}{x}. \) Thus \( \lim_{h \to 0} \frac{\ln(3 + h) - \ln(3)}{h} = f'(3) = \frac{1}{3}. \)