CHAPTER 21

Derivatives of Trig Functions

In Part 3 we have introduced the idea of a derivative of a function, which we defined in terms of a limit. Then we began the task of finding rules that compute derivatives without limits. Here is our list of rules so far.

Constant function rule: $D_x\Big[c\Big] = 0$

Identity function rule: $D_x\Big[x\Big] = 1$

Power rule: $D_x\Big[x^n\Big] = nx^{n-1}$

Exponential rule: $D_x\Big[e^x\Big] = e^x$

Constant multiple rule: $D_x\Big[c\,f(x)\Big] = c\,f'(x)$

Sum-difference rule: $D_x\Big[f(x) \pm g(x)\Big] = f'(x) \pm g'(x)$

Product rule: $D_x\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x)$

Quotient rule: $D_x\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

In this chapter we will expand this list by adding six new rules for the derivatives of the six trigonometric functions:

$$D_x\Big[\sin(x)\Big] \quad D_x\Big[\tan(x)\Big] \quad D_x\Big[\sec(x)\Big] \quad D_x\Big[\cos(x)\Big] \quad D_x\Big[\csc(x)\Big] \quad D_x\Big[\cot(x)\Big]$$

This will require a few ingredients. First, we will need the addition formulas for sine and cosine (Equations 3.12 and 3.13 on page 46):

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\end{aligned}$$

Recall also from Chapter 3 the fundamental identity $\sin^2(x)+\cos^2(x) = 1$.

And we will need these limits from Theorem 10.2 (Chapter 10, page 152):

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}=0 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\sin(h)}{h}=1.$$

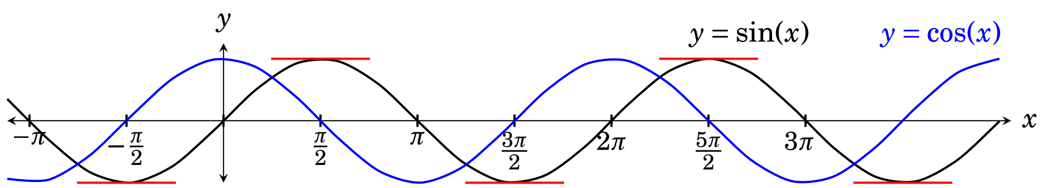
Let's start by computing the derivative of $f(x)=\sin(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{(Definition 16.1)}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} && (f(x)=\sin(x))\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} && \text{(addition formula for sin)}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h} && \text{(regroup)}\\
&= \lim_{h\to 0}\frac{\sin(x)\big(\cos(h)-1\big)\;+\;\cos(x)\sin(h)}{h} && \text{(factor out } \sin(x))\\
&= \lim_{h\to 0}\left(\frac{\sin(x)\big(\cos(h)-1\big)}{h}+\frac{\cos(x)\sin(h)}{h}\right) && \text{(break up fraction)}\\
&= \lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right)\\
&= \lim_{h\to 0}\sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\lim_{h\to 0}\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} && \text{(limit laws)}\\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 && \text{(Theorem 10.2)}\\
&= \cos(x).
\end{aligned}$$

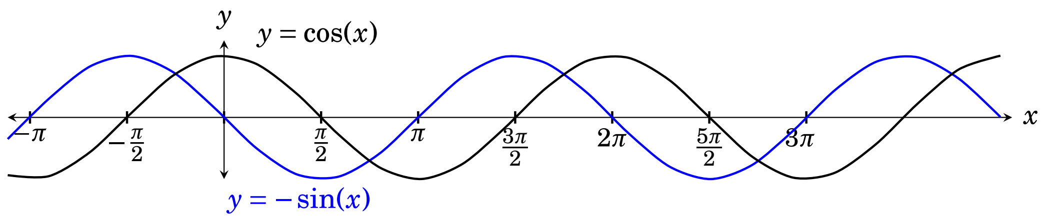
Therefore the derivative of $\sin(x)$ is $\cos(x)$. This is our latest derivative rule.

Rule 9	$D_x\big[\sin(x)\big] \;=\; \cos(x)$

This rule makes sense when we compare the graph of $\sin(x)$ with its derivative $\cos(x)$. The tangent to $\sin(x)$ has slope 0 at integer multiples of $\frac{\pi}{2}$, and these are exactly the places that $\cos(x)=0$. And notice that where the tangent to $\sin(x)$ has positive slope, $\cos(x)$ is positive; where the tangent to $\sin(x)$ has negative slope, $\cos(x)$ is negative. As the derivative of $\sin(x)$, $\cos(x)$ equals the slope of the tangent to $\sin(x)$ at $(x,\sin(x))$.



So what is the derivative of $\cos(x)$? Since $D_x[\sin(x)] = \cos(x)$, you might first guess that $D_x[\cos(x)] = \sin(x)$. But this is not quite right because for $0 < x < \pi$ the tangents to $\cos(x)$ have negative slope, while $\sin(x)$ is positive. However, the graphs below suggest $D_x[\cos(x)] = -\sin(x)$.



In fact this turns out to be exactly right. This chapter's Exercise 19 asks you to adapt the computation on the previous page to get the following rule.

Rule 10 $D_x\Big[\cos(x)\Big] = -\sin(x)$

We now have derivative rules for sin and cos. Next, let's compute the derivative of $\tan(x)$. We will use the rules we just derived in conjunction with the quotient rule and familiar identities.

$$\begin{aligned}
D_x\Big[\tan(x)\Big] &= D_x\left[\frac{\sin(x)}{\cos(x)}\right] \\
&= \frac{D_x\Big[\sin(x)\Big]\cos(x) - \sin(x)D_x\Big[\cos(x)\Big]}{\cos^2(x)} \\
&= \frac{\cos(x)\cos(x) - \sin(x)\big(-\sin(x)\big)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)^2 = \sec^2(x)
\end{aligned}$$

We have a new rule: **Rule 11** $D_x\Big[\tan(x)\Big] = \sec^2(x)$

Exercise 20 asks you to do a similar computation to show that

Rule 12 $D_x\Big[\cot(x)\Big] = -\csc^2(x)$

These two latest formulas fit the shapes of the graphs of tan and cot as suggested by Figure 21.1.

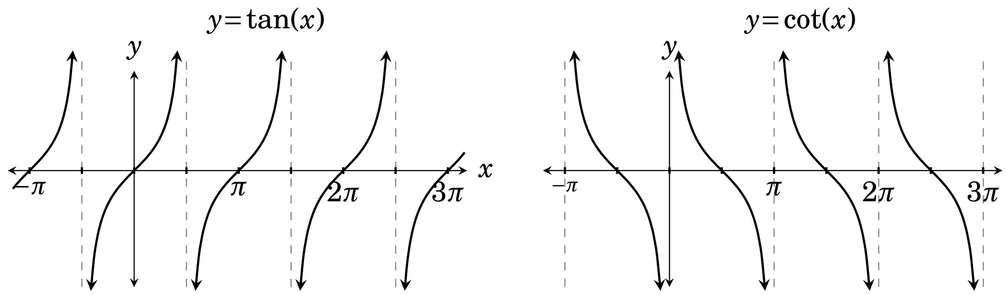


Figure 21.1. Any tangent line to the graph of $y = \tan(x)$ has positive slope. Indeed the slope of the tangent at x is the positive number $y' = \sec^2(x)$. Any tangent line to the graph of $y = \cot(x)$ has negative slope; the slope of the tangent at x is the negative number $y' = -\csc^2(x)$.

There are just two more trig functions to consider: sec and csc. We have

$$\begin{aligned} D_x\Big[\sec(x)\Big] = D_x\left[\frac{1}{\cos(x)}\right] &= \frac{D_x\Big[1\Big]\cdot\cos(x) - 1\cdot D_x\Big[\cos(x)\Big]}{\cos^2(x)} \\ &= \frac{0\cdot\cos(x) - 1\cdot\big(-\sin(x)\big)}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x). \end{aligned}$$

This is our latest rule. **Rule 13** $D_x\Big[\sec(x)\Big] = \sec(x)\tan(x)$

This chapter's Exercise 21 asks you to do a similar computation to prove

Rule 14 $D_x\Big[\csc(x)\Big] = -\csc(x)\cot(x)$.

We now have derivative rules for all six trig functions, which was this chapter's goal. Here is a summary of what we've discovered.

Derivatives of Trig Functions

$D_x\Big[\sin(x)\Big] = \cos(x)$	$D_x\Big[\tan(x)\Big] = \sec^2(x)$	$D_x\Big[\sec(x)\Big] = \sec(x)\tan(x)$
$D_x\Big[\cos(x)\Big] = -\sin(x)$	$D_x\Big[\cot(x)\Big] = -\csc^2(x)$	$D_x\Big[\csc(x)\Big] = -\csc(x)\cot(x)$

Example 21.1 Find the derivative of $y = \dfrac{\sin(x)}{x^2+1}$.

This is a quotient, so we use the quotient rule combined with our new rule for the derivative of sin.

$$
\begin{aligned}
D_x\left[\frac{\sin(x)}{x^2+1}\right] &= \frac{D_x\Big[\sin(x)\Big]\left(x^2+1\right) - \sin(x)D_x\Big[x^2+1\Big]}{\left(x^2+1\right)^2} \\
&= \boxed{\frac{\cos(x)\left(x^2+1\right) - \sin(x)2x}{\left(x^2+1\right)^2}}
\end{aligned}
$$

Example 21.2 Find the derivative of $x^2 + x^3\tan(x) + \pi$.

This is the sum of a power, a product and a constant, so we begin with the sum-difference rule, breaking the problem into three separate derivatives, then using applicable rules.

$$
\begin{aligned}
D_x\Big[x^2 + x^3\tan(x) + \pi\Big] &= D_x\Big[x^2\Big] + D_x\Big[x^3\tan(x)\Big] + D_x\Big[\pi\Big] \\
&= 2x + \underbrace{D_x\Big[x^3\Big]\tan(x) + x^3 D_x\Big[\tan(x)\Big]}_{\text{product rule}} + 0 \\
&= \boxed{2x + 3x^2\tan(x) + x^3\sec^2(x)}
\end{aligned}
$$

With practice you will quickly reach the point where you will do such a problem in your head, in one step. (You may already be there.)

Example 21.3 If $z = \dfrac{e^w\sec(w)}{w}$, find the derivative $\dfrac{dz}{dw}$.

This is a quotient, so our first step is to apply the quotient rule.

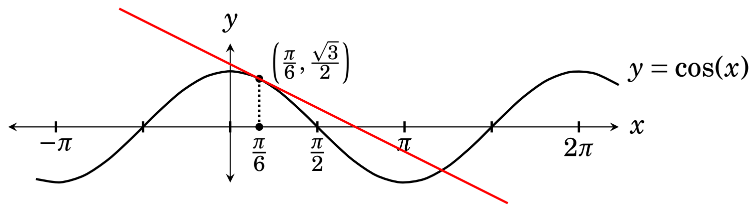
$$
\frac{dz}{dw} = \frac{D_w\left[e^w\sec(w)\right]\cdot w + e^w\sec(w)\cdot D_w\left[w\right]}{w^2}
$$

This now involves $D_w\left[e^w\sec(w)\right]$, and *that* requires the *product* rule.

$$
\begin{aligned}
&= \frac{\left(D_w\left[e^w\right]\sec(w) + e^w D_w\left[\sec(w)\right]\right)\cdot w + e^w\sec(w)\cdot 1}{w^2} \\
&= \frac{\left(e^w\sec(w) + e^w\sec(w)\tan(w)\right)\cdot w + e^w\sec(w)}{w^2} \\
&= \boxed{\frac{e^w\sec(w)\left(w + w\tan(w) + 1\right)}{w^2}}
\end{aligned}
$$

Example 21.4 Find the equation of the tangent line to the graph of $y = \cos(x)$ at the point $\left(\frac{\pi}{6}, \cos\left(\frac{\pi}{6}\right)\right)$.

The slope of the tangent line at the point $(x, \cos(x))$ is given by the derivative $\frac{dy}{dx} = -\sin(x)$. In this problem we are interested in the tangent line at the exact point $\left(\frac{\pi}{6}, \cos\left(\frac{\pi}{6}\right)\right) = \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$, so that tangent line has slope $-\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.



So we are looking for the equation of the line through the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$, with slope $-\frac{1}{2}$. We can get this with the point-slope formula for a line.

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - \frac{\sqrt{3}}{2} &= -\frac{1}{2}\left(x - \frac{\pi}{6}\right) \\ y &= -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2} \end{aligned}$$

Answer: The equation of the tangent line is $\boxed{y = -\frac{1}{2}x + \frac{\pi + 6\sqrt{3}}{12}.}$ ✍

Exercises for Chapter 21

In exercises 1–14 find the derivative of the indicated function.

1. $y = \sqrt[3]{x}\sin(x)$

2. $f(r) = 5r - \cos(r) + \frac{1}{r}$

3. $f(z) = \sin^2(z)$

4. $y = x^4\tan(x)$

5. $y = x^2\sec(x)$

6. $f(r) = 3e^r - \frac{1}{r^2} + \sin(r)$

7. $y = \tan(x) + \frac{1}{x^2} + e^2 + 3$

8. $f(\theta) = 5\theta - \cot(\theta) + \sqrt{\theta}$

9. $f(s) = \tan(s) - \frac{3}{s^2} + 2e^s$

10. $y = \tan^2(x)$

11. $y = \frac{\sqrt{x}\cos(x)}{x^3 + 1}$

12. $y = \frac{x\sin(x)}{e^x}$

13. $y = \frac{x^2 + 5}{x + \sec(x)}$

14. $y = \frac{x\cos(x)}{\sin(x) + 1}$

15. Find $\frac{dz}{dw}$ if $z = \frac{5}{w} + \frac{\tan(w)}{w+1}$.

16. Find $\frac{dz}{dw}$ if $z = \sqrt{w} + 5(w+1)\sec(w)$.

17. Find $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3}+h\right) - \sin\left(\frac{\pi}{3}\right)}{h}$

18. Find $\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right) - \tan\left(\frac{\pi}{4}\right)}{h}$

19. Adapt this chapter's derivation of the rule $D_x\left[\sin(x)\right] = \cos(x)$ to show that $D_x\left[\cos(x)\right] = -\sin(x)$.

20. Adapt this chapter's derivation of the rule $D_x\left[\tan(x)\right] = \sec^2(x)$ to show that $D_x\left[\cot(x)\right] = -\csc^2(x)$.

21. Adapt this chapter's derivation of the rule $D_x\left[\sec(x)\right] = \sec(x)\tan(x)$ to show that $D_x\left[\csc(x)\right] = -\csc(x)\cot(x)$.

22. Suppose $f(x) = (x^2 - \pi^2)\cos(x)$. Find the equation of the tangent line to the graph of $f(x)$ at the point $(\pi, f(\pi))$.

23. Suppose $f(x) = \frac{\sin(x)}{x}$. Find the equation of the tangent line to the graph of $f(x)$ at the point $(\pi, f(\pi))$.

24. Suppose $f(x) = x^3 - x + 2$. Find the equation of the line tangent to the graph of $f(x)$ at the point $(2, 8)$.

25. Find the equation of the tangent line to the graph of $y = \sin(x)$ at the point where $x = \pi$.

26. Find all values of x for which the tangents the graphs of $y = \sqrt{3}\sin(x)$ and $y = -\cos(x)$ are parallel.

27. Find all values of x for which the tangents the graphs of $y = x\sin(x)$ and $y = x^2/4 - \cos(x)$ are parallel.

Exercise Solutions for Chapter 21

1. $y = \sqrt[3]{x}\sin(x) = x^{1/3}\sin(x)$ $\quad y' = \frac{1}{3}x^{1/3-1}\sin(x) + x^{1/3}\cos(x) = \frac{\sin(x)}{3\sqrt[3]{x^2}} + \sqrt[3]{x}\cos(x)$

3. $f(z) = \sin^2(z) = \sin(z)\sin(z)$ $\quad f'(z) = \cos(z)\sin(z) + \sin(z)\cos(z) = 2\sin(z)\cos(z)$

5. $y = x^2\sec(x)$ By product rule: $y' = 2x\sec(x) + x^2\sec(x)\tan(x)$

7. $y = \tan(x) + \frac{1}{x^2} + e^2 + 3 = \tan(x) + x^{-2} + e^2 + 3$ $\quad y' = \sec^2(x) - 2x^{-3} + 0 + 0 = \sec^2(x) - \frac{2}{x^3}$

9. $f(s) = \tan(s) - \frac{3}{s^2} + 2e^s$ $\quad f'(s) = \sec^2(s) + \frac{6}{s^3} + 2e^s$

11. $y = \frac{\sqrt{x}\cos(x)}{x^3+1} = \frac{x^{1/2}\cos(x)}{x^3+1}$

$$\begin{aligned} y' &= \frac{D_x\left[x^{1/2}\cos(x)\right](x^3+1) - x^{1/2}\cos(x)\cdot D_x\left[x^3+1\right]}{(x^3+1)^2} \\ &= \frac{\left(\frac{1}{2}x^{-1/2}\cos(x) + x^{1/2}(-\sin(x))\right)(x^3+1) - x^{1/2}\cos(x)3x^2}{(x^3+1)^2} \\ &= \frac{\left(\frac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x)\right)(x^3+1) - \sqrt{x}\cos(x)3x^2}{(x^3+1)^2} \end{aligned}$$

13. $y = \frac{x^2+5}{x+\sec(x)}$ $\quad y' = \frac{2x\big(x+\sec(x)\big) - \big(x^2+5\big)\big(1+\sec(x)\tan(x)\big)}{\big(x+\sec(x)\big)^2}$

15. $z = \frac{5}{w} + \frac{\tan(w)}{w+1} = 5w^{-1} + \frac{\tan(w)}{w+1}$.

Answer: $\frac{dz}{dw} = -5w^{-2} + \frac{\sec^2(w)(w+1) - \tan(w)\cdot 1}{(w+1)^2} = -\frac{5}{w^2} + \frac{\sec^2(w)(w+1) - \tan(w)}{(w+1)^2}$

17. Find $\lim_{h\to 0}\frac{\sin(\pi/3+h) - \sin(\pi/3)}{h}$. Let $f(x) = \sin(x)$. Then by the definition of the derivative, this limit equals $f'(\pi/3) = \cos(\pi/3) = \frac{1}{2}$.

19. Adapt the derivation of the rule $D_x\big[\sin(x)\big] = \cos(x)$ to prove $D_x\big[\cos(x)\big] = -\sin(x)$.

$$\begin{aligned}
D_x\Big[\cos(x)\Big] &= \lim_{h\to 0}\frac{\cos(x+h) - \cos(x)}{h} && \text{(Definition 16.1)} \\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} && \text{(addition formula for cos)} \\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h) - \cos(x) - \sin(x)\sin(h)}{h} && \text{(regroup)} \\
&= \lim_{h\to 0}\frac{\cos(x)\big(\cos(h)-1\big) + \sin(x)\sin(h)}{h} && \text{(factor out } \sin(x)\text{)} \\
&= \lim_{h\to 0}\left(\cos(x)\frac{\cos(h)-1}{h} - \sin(x)\frac{\sin(h)}{h}\right) && \text{(break up fraction)} \\
&= \lim_{h\to 0}\cos(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h} - \lim_{h\to 0}\sin(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} && \text{(limit laws)} \\
&= \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x). && \text{(Theorem 10.2)}
\end{aligned}$$

21. Adapt this chapter's derivation of the rule $D_x[\sec(x)] = \sec(x)\tan(x)$ to show that $D_x[\csc(x)] = -\csc(x)\cot(x)$.

$$D_x\Big[\csc(x)\Big] = D_x\left[\frac{1}{\sin(x)}\right] = \frac{D_x\Big[1\Big]\cdot\sin(x) - 1\cdot D_x\Big[\sin(x)\Big]}{\sin^2(x)} = \frac{0\cdot\sin(x) - 1\cdot\big(\cos(x)\big)}{\sin^2(x)}$$

$$= \frac{-\cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x).$$

23. Find the equation of the tangent line to the graph of $f(x) = \dfrac{\sin(x)}{x}$ at $(\pi, f(\pi))$.

The slope of the line is given by the derivative $f'(x) = \dfrac{\cos(x)x - \sin(x)\cdot 1}{x^2}$. The line passes through the point $(\pi, f(\pi)) = (\pi, 0)$. At this point the slope of the tangent line is $m = f'(\pi) = \dfrac{\cos(\pi)\pi - \sin(\pi)}{\pi^2} = \dfrac{-1\cdot\pi - 0}{\pi^2} = -\dfrac{1}{\pi}$. We can get the equation of the line by the point-slope formula:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 0 &= -\frac{1}{\pi}(x - \pi) \\ \textbf{Answer:} \quad y &= -\frac{1}{\pi}x + 1. \end{aligned}$$

25. Find the equation of the tangent line to the graph of $y = \sin(x)$ at the point $x = \pi$.

We are looking for the tangent at the point $(\pi, \sin(\pi)) = (\pi, 0)$. The slope of the line is given by the derivative $\dfrac{dy}{dx} = \cos(x)$. At $(\pi, 0)$ the slope of the tangent line is $m = \dfrac{dy}{dx}\Big|_{x=\pi} = \cos(\pi) = -1$. The point-slope formula gives the line's equation:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 0 &= --1\cdot(x - \pi) \\ \textbf{Answer:} \quad y &= -x + \pi. \end{aligned}$$