CHAPTER 13

Limits at Infinity

Until now we've been dealing with limits of the form $\lim_{x\to c} f(x)$, where x approaches a fixed number c. In this chapter we'll make sense of a limits like $\lim_{x\to\infty} f(x)$, where x approaches ∞ instead of a finite number c.

Consider the function $f(x)$ graphed in Figure 13.1. Notice that $f(x) > 2$ as long as x is bigger than about 1. But as x becomes bigger and bigger, moving toward ∞, the corresponding $f(x)$ value gets closer and closer to 2. We express this in symbols as $\lim_{x\to\infty} f(x) = 2$.

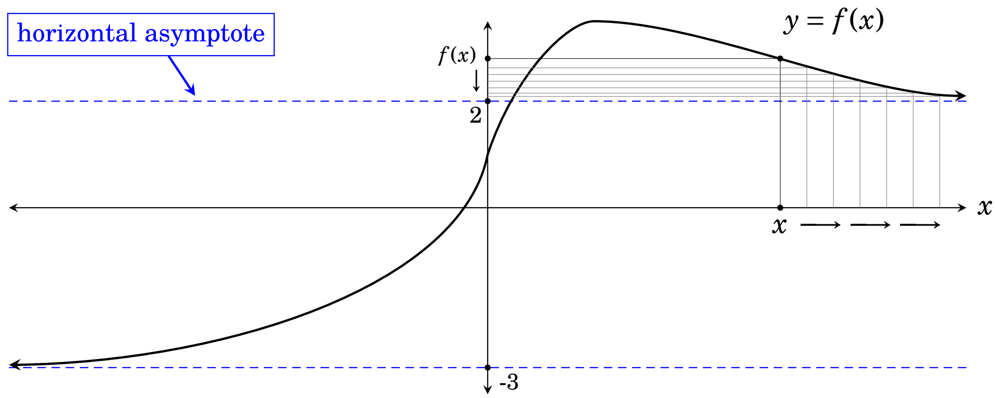


Figure 13.1. A function for which $\lim_{x\to\infty} f(x) = 2$. The line $y = 2$ is a horizontal asymptote.

Notice that the graph gets closer and closer to the dashed horizontal line $y = 2$ as x moves towards ∞. This line is called a **horizontal asymptote** of the function $f(x)$. It is not really a part of the graph, but it helps us visualize the behavior of $f(x)$ as x gets bigger and bigger.

Figure 13.2 shows what happens to this same function as x becomes increasingly big in the *negative* direction, moving toward $-\infty$. As this happens the corresponding $f(x)$ approaches -3. We express this as $\lim_{x\to-\infty} f(x) = -3$. As x approaches $-\infty$, the graph of $f(x)$ becomes ever closer to the horizontal line $y = -3$, which is a second horizontal asymptote of $f(x)$.

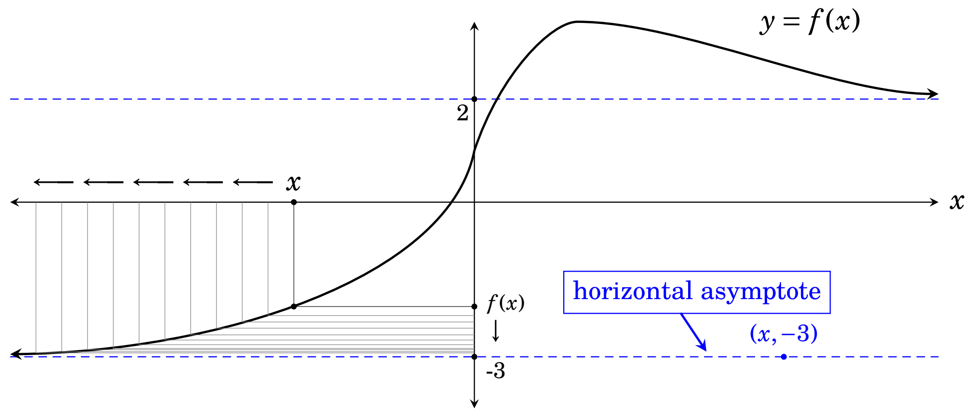


Figure 13.2. A function for which $\lim_{x \to -\infty} f(x) = -3$. The line $y = 2$ is a horizontal asymptote.

Limits like $\lim_{x \to \infty} f(x) = 2$ and $\lim_{x \to -\infty} f(x) = -3$, where x approaches ∞ or $-\infty$ are called **limits at infinity**, which is a bit of a misnomer because x is never "at" ∞ (or $-\infty$), just approaching it. Here is an informal definition.

Definition 13.1 Limits at infinity

- $\lim_{x \to \infty} f(x) = L$ means that x approaching ∞ causes $f(x)$ to approach the number L. In such a case the line $y = L$ is a horizontal asymptote.
- $\lim_{x \to -\infty} f(x) = M$ means that x approaching $-\infty$ causes $f(x)$ to approach the number M. In such a case the line $y = M$ is a horizontal asymptote.

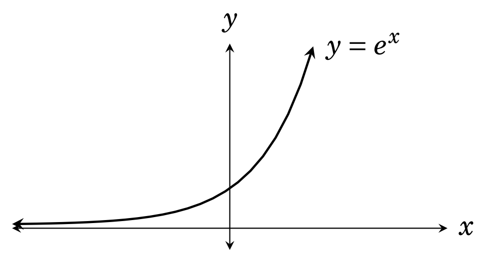
We sometimes say that the limits $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = M$ describe the **end behavior** of the function f, as they tell what happens to $f(x)$ at the two "ends" of the number line ∞ and $-\infty$.

This chapter introduces some methods for computing limits at infinity and finding horizontal asymptotes. (That is, for determining the end behavior of functions.) These techniques will not apply to every function, but they will be sufficient for most purposes in this course. (Chapter 37 will develop more powerful techniques.) But before introducing our new techniques, let's look at some easy examples to solidify our understanding of the concepts.

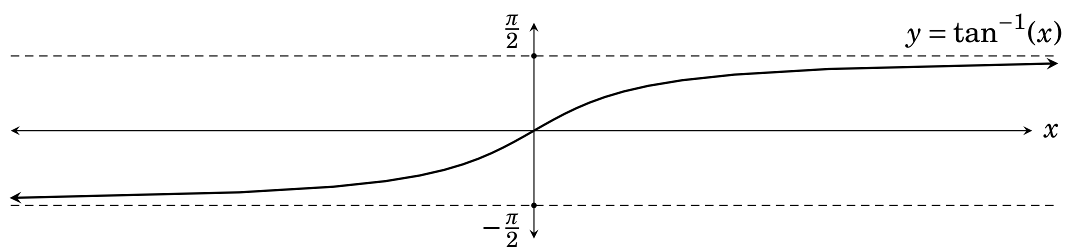
13.1 End Behavior of Functions

This section examines the end behavior of a handful of familiar functions. In each case we'll determine the end behavior of f by examining $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$, limits that will be automatic because of our familiarity with the functions.

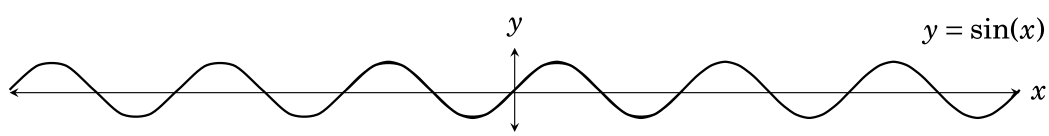
First consider $f(x) = e^x$, graphed below. First, $\lim_{x\to-\infty} e^x = 0$, so the x-axis $y = 0$ is a horizontal asymptote. But as x gets bigger in the *positive* direction, e^x gets bigger without bound. We express this by writing $\lim_{x\to\infty} e^x = \infty$



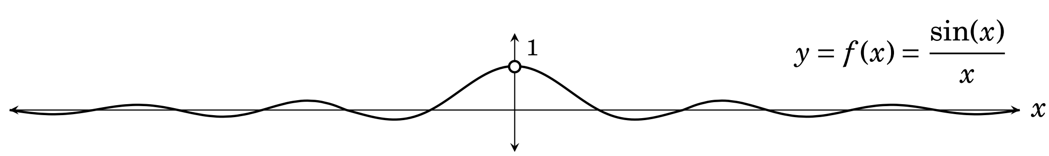
Now consider the function $\tan^{-1}(x)$. We have $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$ and $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$. (If this is not obvious, please review $\tan^{-1}$ in Chapter 6.) This function has two horizontal asymptotes, the lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.



The function $\sin(x)$ just oscillates forever as x goes to ∞ or $-\infty$, neither approaching infinity nor any number. Thus $\lim_{x\to-\infty} \sin(x)$ and $\lim_{x\to\infty} \sin(x)$ DNE.

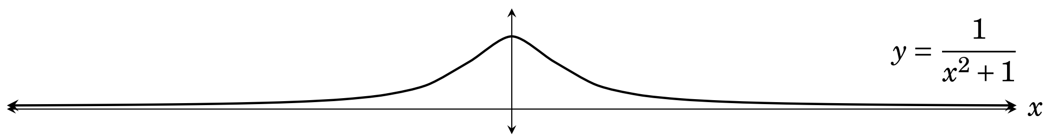


But now modify $\sin(x)$ by multiplying it by $\frac{1}{x}$ to get $f(x) = \sin(x)\frac{1}{x} = \frac{\sin(x)}{x}$. If x is very large, then its reciprocal $\frac{1}{x}$ is very small, so the factor of $\frac{1}{x}$ scales $\sin(x)$ down to a very small number, compressing the graph close to the x-axis, as shown below. We have $\lim_{x\to-\infty} \frac{\sin(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$. The x-axis is a horizontal asymptote.

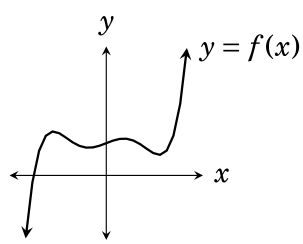


(Since $f(0)$ is not defined, but $\lim_{x\to0} f(x) = 1$, the graph has a hole at $(0,1)$.)

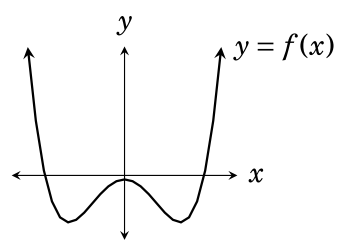
For our next example, consider the function $y = \dfrac{1}{x^2+1}$. When x is big, the denominator x^2+1 is even bigger, so its reciprocal becomes closer and closer to zero as x gets bigger and bigger. Thus $\lim\limits_{x\to-\infty} \dfrac{1}{x^2+1} = 0$ and $\lim\limits_{x\to\infty} \dfrac{1}{x^2+1} = 0$, so the x-axis is a horizontal asymptote.



Our final example of this section is a bit more general. We will examine the end behavior of a polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ of degree n. From your experience working with polynomials in previous courses, you know that the shape of these graphs is influenced by whether the degree n is even or odd, and whether the coefficient coefficient a_n of x^n is positive or negative. The diagrams below illustrate the difference with a_n positive. (Making a_n negative turns the graphs upside down.)



Odd-degree polynomial ($a_n > 0$)



Even-degree polynomial ($a_n > 0$)

Assuming a_n is positive (as in the above graphs) we see $\lim\limits_{x\to\infty} f(x) = \infty$ whether n is even or odd. Also $\lim\limits_{x\to-\infty} f(x) = -\infty$ if n is odd, but $\lim\limits_{x\to-\infty} f(x) = \infty$ if n is even. This sounds complicated when written down, but the idea is very simple. Here is the summary.

Fact 13.1 (End behavior of polynomials)
Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial of positive degree n.

$$\text{If } n \text{ is even, then } \begin{cases} \lim\limits_{x\to\infty} f(x) = \begin{cases} \infty & \text{if } a_n \text{ is positive} \\ -\infty & \text{if } a_n \text{ is negative} \end{cases} \\ \lim\limits_{x\to-\infty} f(x) = \begin{cases} \infty & \text{if } a_n \text{ is positive} \\ -\infty & \text{if } a_n \text{ is negative} \end{cases} \end{cases}$$

$$\text{If } n \text{ is odd, then } \begin{cases} \lim\limits_{x\to\infty} f(x) = \begin{cases} \infty & \text{if } a_n \text{ is positive} \\ -\infty & \text{if } a_n \text{ is negative} \end{cases} \\ \lim\limits_{x\to-\infty} f(x) = \begin{cases} -\infty & \text{if } a_n \text{ is positive} \\ \infty & \text{if } a_n \text{ is negative} \end{cases} \end{cases}$$

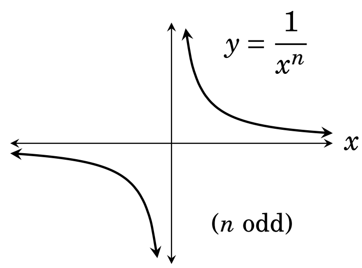
13.2 End Behavior of Rational Functions

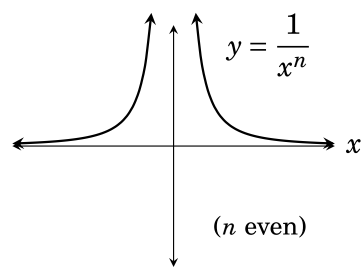
Recall that a **rational function** is a function of form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. For example, $f(x) = \frac{3x+1}{x^2+x-4}$ is a rational function. Technically, any polynomial $p(x)$ is also a rational function, as $p(x) = \frac{p(x)}{1}$ (note that the denominator 1 is a polynomial, as $1 = 0x+1$). Likewise functions such as $f(x) = \frac{1}{x^2+x-4}$ and $g(x) = \frac{1}{x^4}$ are rational functions.

This section is concerned with determining the end behavior of rational functions, that is, determining $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)}$ and $\lim\limits_{x\to-\infty} \dfrac{p(x)}{q(x)}$ where p and q are polynomials. The key to doing this is exploiting the following fact.

Fact 13.2 If n is a positive integer, then $\lim\limits_{x\to\infty} \dfrac{1}{x^n} = 0$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x^n} = 0.$

Both of these limits equal 0 because denominator x^n grows without bound as $x \to \pm\infty$, so its reciprocal shrinks to 0. We can also glean this by looking at the graph of $y = \frac{1}{x^n}$ (with should be familiar to you), shown below. The graphs look different depending on whether n is even or odd, but in each case the x-axis $y = 0$ is plainly a horizontal asymptote, so $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^n} = 0.$





Knowing Fact 13.2 is key to determining other limits at infinity.

Example 13.1 $\lim\limits_{x\to\infty}\left(3+\dfrac{2}{x^4}\right) = \lim\limits_{x\to\infty} 3 + \lim\limits_{x\to\infty}\dfrac{2}{x^4} = \lim\limits_{x\to\infty} 3 + 2\lim\limits_{x\to\infty}\dfrac{1}{x^4} = 3+2\cdot 0 = \boxed{3}.$

Here we used several limit laws combined with the fact that $\lim\limits_{x\to\infty}\frac{1}{x^4} = 0$. Of course you'll probably skip steps while working such problems, and possibly use Fact 13.2 multiple times in the same problem, as in our next example.

Example 13.2 $\lim\limits_{x\to\infty} \dfrac{5+\dfrac{1}{x}+\dfrac{3}{x^2}}{3-\dfrac{2}{x}+\dfrac{1}{x^3}} = \dfrac{5+0+0}{3-0+0} = \boxed{\dfrac{5}{3}}.$

Now we introduce a useful trick for finding limits at infinity $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ of *any* rational function. This trick sometimes works for other types of functions, as we'll see. The idea is to look for the highest power x^n that occurs in the denominator $q(x)$, and then multiply by 1 in the form of

$$1 = \frac{\frac{1}{x^n}}{\frac{1}{x^n}}.$$

Once this is distributed into the top and bottom, you will get a form to which Fact 13.2 applies. A few examples should suffice to illustrate this.

Example 13.3 Find $\lim_{x\to\infty} \frac{5x^3+x^2+2x}{3x^3-2x^2+1}$ and $\lim_{x\to-\infty} \frac{5x^3+x^2+2x}{3x^3-2x^2+1}$.

Here the highest power of x in the denominator is x^3, so we multiply the limit by $\frac{1}{x^3}$ over itself. Then we distribute into the top and bottom, and use Fact 13.2 wherever it applies:

$$\lim_{x\to\infty} \frac{5x^3+x^2+2x}{3x^3-2x^2+1} = \lim_{x\to\infty} \frac{5x^3+x^2+2x}{3x^3-2x^2+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{5+\frac{1}{x}+\frac{2}{x^2}}{3-\frac{2}{x}+\frac{1}{x^3}} = \frac{5+0+0}{3-0+0} = \boxed{\frac{5}{3}}.$$

Using the exact same steps, we can compute the limit as x goes to negative infinity: $\lim_{x\to-\infty} \frac{5x^3+x^2+x}{3x^3-2x^2+1} = \boxed{\frac{5}{3}}$. ✍

Example 13.4 Find $\lim_{x\to\infty} \frac{5x^2+x+2}{3x^3-2x^2+1}$.

This is like the previous example except that the powers on the numerator are lower. But again the highest power of x in the denominator is x^3, so we multiply by $\frac{1}{x^3}$ over itself:

$$\lim_{x\to\infty} \frac{5x^2+x+2}{3x^3-2x^2+1} = \lim_{x\to\infty} \frac{5x^2+x+2}{3x^3-2x^2+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{5}{x}+\frac{1}{x^2}+\frac{2}{x^3}}{3-\frac{2}{x}+\frac{1}{x^3}} = \frac{0+0+0}{3-0+0} = \frac{0}{3} = \boxed{0}.$$ ✍

The next example illustrates that a limit at infinity can be an infinite limit, that is, it can equal ∞ or $-\infty$.

Example 13.5 Find $\lim\limits_{x\to\infty} \dfrac{5x^4+x+2}{3x^3-2x^2+1}$ and $\lim\limits_{x\to-\infty} \dfrac{5x^4+x+2}{3x^3-2x^2+1}$.

The highest power of x in the bottom is x^3, so we multiply by $\frac{1}{x^3}$ over itself:

$$\lim_{x\to\infty} \frac{5x^4+x+2}{3x^3-2x^2+1} = \lim_{x\to\infty} \frac{5x^4+x+2}{3x^3-2x^2+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{5x+\frac{1}{x^2}+\frac{2}{x^3}}{3-\frac{2}{x}+\frac{1}{x^3}}.$$

In this limit the fractional terms $\frac{1}{x^2}$, $\frac{2}{x^3}$, $\frac{2}{x}$ and $\frac{1}{x^3}$ all approach 0, so the denominator approaches $3+0+0=3$. However the $5x$ on the numerator goes to $+\infty$, so we conclude $\lim\limits_{x\to\infty} \dfrac{5x^4+x+2}{3x^3-2x^2+1} = \infty$.

Next we find the limit as x goes to $-\infty$. Using exactly the same steps,

$$\lim_{x\to-\infty} \frac{5x^4+x+2}{3x^3-2x^2+1} = \lim_{x\to-\infty} \frac{5x^4+x+2}{3x^3-2x^2+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to-\infty} \frac{5x+\frac{1}{x^2}+\frac{2}{x^3}}{3-\frac{2}{x}+\frac{1}{x^3}}.$$

This time the $5x$ on top goes to $-\infty$ while the denominator approaches 3. Thus we conclude $\lim\limits_{x\to-\infty} \dfrac{5x^4+x+2}{3x^3-2x^2+1} = -\infty$. ✍

After working a few exercises like the examples above, you will begin to see some patterns that lead to shortcuts. For one, in working out a limit at infinity of a rational function in which the highest powers on the top and bottom are *the same*, the lower powers get knocked out, and the answer will be the fraction of the coefficients of the highest powers. For example,

$$\lim_{x\to\infty} \frac{\mathbf{7}x^3+3x^2+x}{\mathbf{4}x^3-3x^2+3} = \frac{7}{4}.$$

If the highest power occurs *only on the bottom*, the limit is 0. Example:

$$\lim_{x\to\infty} \frac{7x^3+3x^3+x}{4x^5-3x^2+3} = 0.$$

You'd expect this because as $x\to\infty$, the higher-powered bottom becomes vastly larger than the top, so the fraction becomes smaller and smaller.

Finally, if the highest power occurs *only on the top*, the limit will be ∞ or $-\infty$, and you can reason it out as in the above example. For instance,

$$\lim_{x\to-\infty} \frac{7x^5+3x^3+x}{4x^4-3x^2+3} = -\infty.$$

If you are ever in doubt, work it out without the shortcuts.

The next example will indicate how our reciprocal-of-the-highest-power trick can sometimes work for functions that are not rational functions. It will use the following algebraic property.

> **Fact 13.3** (Algebraic property review)
> Suppose a and b are real numbers and b is positive.
>
> - If a is positive, then $a\sqrt{b} = \sqrt{a^2 b}$.
> - If a is negative, then $a\sqrt{b} = -\sqrt{a^2 b}$.

For example, $3\sqrt{4} = \sqrt{3^2 \cdot 4}$ (both sides equal 6) and $(-3)\sqrt{4} = -\sqrt{(-3)^2 \cdot 4}$ (both sides equal -6). You use this property all the time, especially the first part. We highlight it because it's very easy to make a mistake in applying it if a is variable. You may know that x is a negative number but still mistakenly write $x\sqrt{b} = \sqrt{x^2 b}$ instead of the correct $x\sqrt{b} = -\sqrt{x^2 b}$.

Example 13.6 Find $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2+2x+1}}{x-2}$ and $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+2x+1}}{x-2}$.

The highest power of x on the denominator is x^1. Let's do the first limit using our usual trick. In this case x approaches ∞, so x is positive, and so is $\frac{1}{x}$. Below we use the first part of Fact 13.3 to bring $\frac{1}{x}$ inside the radical.

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2x+1}}{x-2} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{\frac{1}{x}} \frac{\sqrt{x^2+2x+1}}{x-2} = \lim_{x\to-\infty} \frac{\sqrt{\left(\frac{1}{x}\right)^2 \left(x^2+2x+1\right)}}{\frac{1}{x}\left(x-2\right)}$$

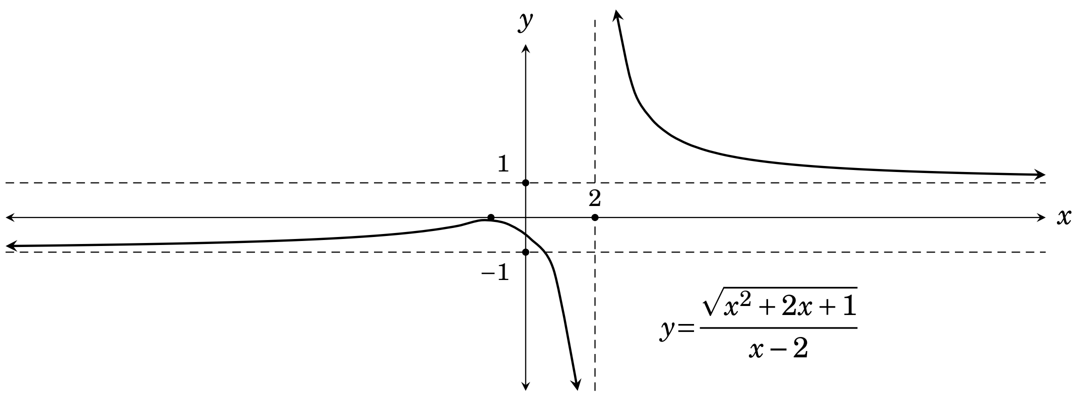
$$= \lim_{x\to-\infty} \frac{\sqrt{1+\frac{2}{x}+\frac{1}{x^2}}}{1-\frac{2}{x}} = \frac{\sqrt{1+0+0}}{1-0} = \boxed{1}$$

But when x approaches $-\infty$, it is *negative*, so $\frac{1}{x}$ is negative too. In applying Fact 13.3, we have to insert a negative sign in front of the radical.

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2x+1}}{x-2} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{\frac{1}{x}} \frac{\sqrt{x^2+2x+1}}{x-2} = \lim_{x\to-\infty} \frac{-\sqrt{\left(\frac{1}{x}\right)^2 \left(x^2+2x+1\right)}}{\frac{1}{x}\left(x-2\right)}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{2}{x}+\frac{1}{x^2}}}{1-\frac{2}{x}} = \frac{-\sqrt{1+0+0}}{1-0} = \boxed{-1}$$

These limits tell us that $f(x) = \dfrac{\sqrt{x^2+2x+1}}{x-2}$ has two horizontal asymptotes, $y = 1$ and $y = -1$. The function and its asymptotes are drawn below. There is also a vertical asymptote at $x = 2$, where the denominator is zero.



Notice that although the graph of a function never touches any of its vertical asymptotes (except in very pathological instances), it is quite possible that it will touch a horizontal asymptote, as happens in this example. ✍

Please note that this sections's technique of multiplying by $\frac{1}{x^n}/\frac{1}{x^n}$ is only for limits **at infinity**, that is, for $x \to \pm\infty$. Other techniques apply in other circumstances, as the next example reminds us.

Example 13.7 Consider the function $f(x) = \dfrac{3x-15}{2x^4-50x^2}$. Find: (a) $\lim\limits_{x\to 0} f(x)$, (b) $\lim\limits_{x\to 1} f(x)$, (c) $\lim\limits_{x\to 5} f(x)$, and (d) $\lim\limits_{x\to\infty} f(x)$.

(a) Here $f(0) = \frac{-15}{0}$ (undefined). This is a nonzero number over zero, so we expect an infinite limit. Looking closer, $\lim\limits_{x\to 0} \dfrac{3x-15}{2x^4-50x^2} = \lim\limits_{x\to 0} \dfrac{3x-15}{x^2(2x^2-50)}$. The numerator approaches −15 (negative) and the denominator approaches 0 *and is negative*. Thus $\lim\limits_{x\to 0} f(x) = \boxed{\infty}$.

(b) When $x \to 1$, the denominator does not approach 0, so we can simply use a familiar limit law: $\lim\limits_{x\to 1} \dfrac{3x-15}{2x^4-50x^2} = \dfrac{3\cdot 1-15}{2\cdot 1^4-50\cdot 1^2} = \dfrac{-12}{-48} = \boxed{\dfrac{1}{4}}$.

(c) In $\lim\limits_{x\to 5} f(x)$, we are getting $\frac{0}{0}$, so try to factor and cancel: $\lim\limits_{x\to 5} \dfrac{3x-15}{2x^4-50x^2} =$
$= \lim\limits_{x\to 5} \dfrac{3(x-5)}{2x^2(x^2-25)} = \lim\limits_{x\to 5} \dfrac{3(x-5)}{2x^2(x-5)(x+5)} = \lim\limits_{x\to 5} \dfrac{3}{2x^2(x+5)} = \dfrac{3}{2\cdot 5^2(5+5)} = \boxed{\dfrac{3}{500}}$.

(d) In $\lim\limits_{x\to\infty} f(x)$, the denominator has the highest power, so by this section's techniques $\lim\limits_{x\to\infty} f(x) = \boxed{0}$. ✍

We close this section with one final technique that works for some limits at infinity. We say that a function $f(x)$ is **bounded** if there are numbers A and B for which $A \le f(x) \le B$ for all x in the domain of f. A bounded function never gets higher than B or lower than A. For example, $\sin(x)$ is bounded because $-1 \le \sin(x) \le 1$. Also $\tan^{-1}(x)$ is bounded because $-\frac{\pi}{2} \le \tan^{-1}(x) \le \frac{\pi}{2}$.

Fact 13.4 (Limit at infinity of a quotient with bounded numerator)

- If $f(x)$ is bounded and $\lim_{x\to\infty} g(x) = \pm\infty$, then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$.
- If $f(x)$ is bounded and $\lim_{x\to-\infty} g(x) = \pm\infty$, then $\lim_{x\to-\infty} \dfrac{f(x)}{g(x)} = 0$.

(We assume here that the domain of f is such that these limits are meaningful, that is, $f(x)$ is defined for sufficiently large positive or negative values of x.)

Fact 13.4 should be pretty much obvious to you at this point. If in $\lim_{x\to\infty} \frac{f(x)}{g(x)}$, the numerator $f(x)$ is bounded but the denominator $g(x)$ is getting bigger and bigger without bound, then the ratio $\frac{f(x)}{g(x)}$ must diminish to zero.

Example 13.8 Find $\lim_{x\to\infty} \dfrac{3+\sin(x)+\cos(\pi x)}{4+\sqrt{x}}$.

Because $-1 \le \sin(x) \le 1$ and $-1 \le \cos(\pi x) \le 1$, it follows that

$$3-1-1 \;\le\; 3+\sin(x)+\cos(\pi x) \;\le\; 3+1+1,$$

which means

$$1 \;\le\; 3+\sin(x)+\cos(\pi x) \;\le\; 5.$$

Therefore the numerator is bounded. And since $\lim_{x\to\infty} 4+\sqrt{x} = \infty$, Fact 13.4 implies that $\lim_{x\to\infty} \dfrac{3+\sin(x)+\cos(\pi x)}{4+\sqrt{x}} = 0$. ✍

Example 13.9 Find $\lim_{x\to\infty} \dfrac{\tan^{-1}(x)}{x}$ and $\lim_{x\to\infty} \dfrac{\sin^{-1}(x)}{x}$

The function $\tan^{-1}$ is bounded because $-\frac{\pi}{2} \le \tan^{-1}(x) \le \frac{\pi}{2}$ for all x. By Fact 13.4, $\lim_{x\to\infty} \dfrac{\tan^{-1}(x)}{x} = 0$.

Certainly also $\sin^{-1}$ is bounded because $-\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$ for all x. But the domain of $\sin^{-1}$ is the closed interval $[-1,1]$, so $\sin^{-1}(x)$ is undefined for any number x beyond 1. Therefore $\lim_{x\to\infty} \dfrac{\sin^{-1}(x)}{x}$ DNE. ✍

13.3 Finding Asymptotes

Some exercises will ask you to find a function's horizontal asymptotes. (Or perhaps you'll want to find them to better understand some function you are working with.) For convenience we restate and highlight the method.

How to find the horizontal asymptotes (if any) of f(x)

- Compute $\lim_{x\to\infty} f(x)$. If you get a finite number L then the line $y = L$ is a horizontal asymptote.
- Compute $\lim_{x\to-\infty} f(x)$. If you get a finite number M then the line $y = M$ is a horizontal asymptote.

Therefore a function can have at most two horizontal asymptotes, one for $x \to \infty$ and another for $x \to -\infty$. Some functions (like x^2) have no horizontal asymptotes at all, and others (like e^x) have only one. By contrast, recall that a function can have arbitrarily many *vertical* asymptotes.

Example 13.10 Find all horizontal and vertical asymptotes of the function $f(x) = \dfrac{3x^2 - 21x + 36}{x^4 - 4x^3 - x^2 + 4x}$.

To find the horizontal asymptotes we need to evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Because f is a rational function with highest power in the denominator, we know (as explained on page 187) that $\lim_{x\to\infty} f(x) = 0$, which means that the line $y = 0$ is a horizontal asymptote. In the negative direction, $\lim_{x\to-\infty} f(x) = 0$ and we get the same horizontal asymptote $y = 0$ Thus f has only one horizontal asymptote, the line $y = 0$ (which is the x-axis).

Now let's find the vertical asymptotes using the guidelines of Section 12.3. Such asymptotes are potentially located at the x values that make the denominator of f zero. To find them, try to factor:

$$
\begin{aligned}
f(x) = \frac{3x^2 - 21x + 36}{x^4 - 4x^3 - x^2 + 4x} = \frac{3\left(x^2 - 7x + 12\right)}{x\left(x^3 - 4x^2 - x + 4\right)} &= \frac{3(x-3)(x-4)}{x(x^3 - 4x^2 - x + 4)} \\
&= \frac{3(x-3)(x-4)}{x\left(x^2(x-4) - (x-4)\right)} \\
&= \frac{3(x-3)(x-4)}{x(x^2-1)(x-4)} \\
&= \frac{3(x-3)(x-4)}{x(x+1)(x-1)(x-4)}
\end{aligned}
$$

The denominator is zero for $x = 0$, $x = -1$, $x = 1$ and $x = 4$. These are our candidates vertical asymptotes.

To see if each $x = c$ is really a vertical asymptote, we need to see if $\lim_{x \to c^{\pm}} f(x) = \pm\infty$. But before doing this, notice that there is some further cancellation in the above expression for $f(x)$. As long as $x \neq 4$ the two factors of $(x-4)$ cancel. and we get

$$f(x) = \frac{3(x-3)}{x(x+1)(x-1)} \qquad \text{(provided } x \neq 4\text{).}$$

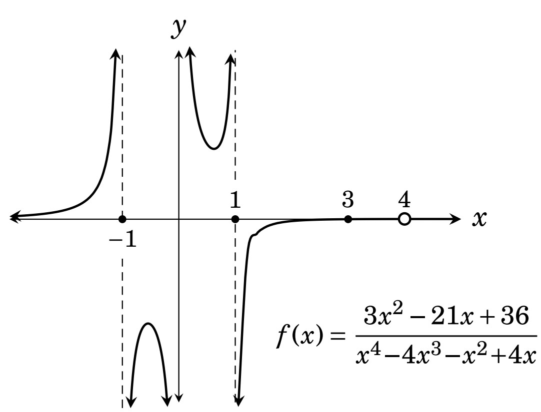
(But note $f(4) = \frac{0}{0}$ is not defined.) Now let's check our asymptote candidates one-by-one. We'll use right-hand limits, though left would work just as well.

Check $x = 0$: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{3(x-3)}{x(x+1)(x-1)} = \infty$ (because top approaches -9 while bottom approaches 0 and is negative). Since we got $-\infty$, the line $x = 0$ is a vertical asymptote.

Check $x = -1$: $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{3(x-3)}{x(x+1)(x-1)} = -\infty$ (because top approaches -12 while bottom approaches 0 and is positive). Since we got ∞, the line $x = -1$ is a vertical asymptote.

Check $x = 1$: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{3(x-3)}{x(x+1)(x-1)} = -\infty$ (because top approaches -6 while bottom approaches 0 and is positive). Since we got $-\infty$, the line $x = -1$ is a vertical asymptote.

Check $x = 4$: $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{3(x-3)}{x(x+1)(x-1)} = \frac{3}{60} = \frac{1}{20}$. Because we didn't get $\pm\infty$, **there is no vertical asymptote at** $x = 4$.



In summary, f has one horizontal asymptote $y = 0$ and three vertical asymptotes, the lines $x = 0$, $x = -1$ and $x = 1$. The graph is sketched above. Notice the asymptotes and the hole in the graph at $(4, \frac{1}{20})$ (where there was no asymptote at $x = 4$). Note also the x-intercept of 3. ✍

Exercises for Chapter 13

Most of the these questions involve limits at infinity. But overall these questions are cumulative, representing all chapters in Part 2 of the text.

1. $\lim_{x\to-\infty} e^{5x-3} =$

2. $\lim_{x\to\infty} e^{1/x} =$

3. $\lim_{x\to-\infty} \tan^{-1}(x^2) =$

4. $\lim_{x\to\infty} \sec^{-1}(x) =$

5. Evaluate the following limits.

(a) $\lim_{x\to0}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right)$

(b) $\lim_{x\to1^+}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right)$

(c) $\lim_{x\to\infty}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right)$

(d) $\lim_{x\to\pi}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right)$

6. Evaluate the following limits.

(a) $\lim_{x\to8}\frac{x-2}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)}$

(b) $\lim_{x\to2}\frac{x-2}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)}$

(c) $\lim_{x\to0^+}\frac{x-2}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)}$

(d) $\lim_{x\to\infty}\frac{x-2}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)}$

7. Answer the following questions about the function $y=f(x)$ graphed below.

(a) $\lim_{x\to\infty} f(x) =$

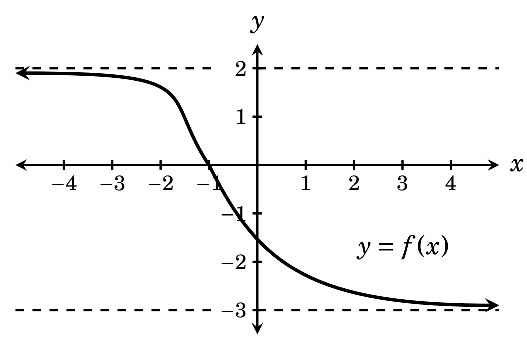
(b) $\lim_{x\to-\infty} f(x) =$

(c) $\lim_{x\to-\infty} (f(x)+4) =$

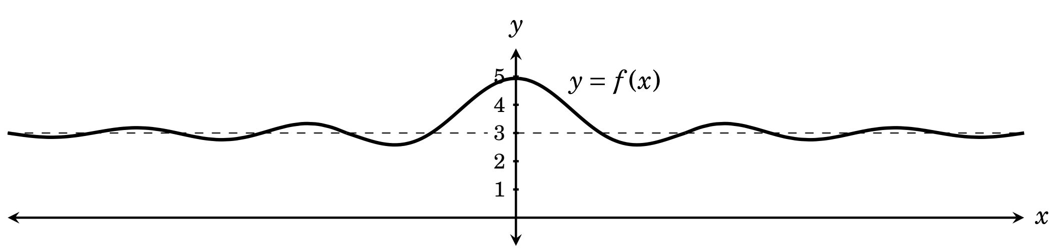
(d) $\lim_{x\to\infty} f\left(\frac{1-x}{x}\right) =$

(e) $\lim_{x\to\infty} \cos\left(\pi f(x)\right) =$

(f) $\lim_{x\to\infty} 2^{f(x)} =$



8. Answer the following questions about the function $y=f(x)$ graphed below.



(a) $\lim_{x\to\infty} f(x) =$

(b) $\lim_{x\to-\infty} f(x) =$

(c) $\lim_{x\to\infty} \ln\left(4-f(x)\right) =$

(d) $\lim_{x\to\infty} f\left(\frac{1}{x}\right) =$

(e) $\lim_{x\to\infty} \frac{1}{f(x)} =$

(f) $\lim_{x\to\infty} \frac{1}{f(1/x)} =$

9. Find all vertical and horizontal asymptotes of $f(x)=\dfrac{x^2-4}{5x^2-10x}$.

10. Find all vertical and horizontal asymptotes of $f(x)=\dfrac{x^2-x-6}{x^2-4x+3}$.

11. Find all vertical and horizontal asymptotes of $f(x)=\dfrac{7x^3-7x^2}{x^2-1}$.

12. Find all vertical and horizontal asymptotes of $f(x)=\dfrac{-x^2-3x+15}{x^2+9x+20}$.

13. Find the horizontal and vertical asymptotes of $f(x)=\dfrac{2x^2-8}{x^2+3x+2}$.

14. Find the horizontal and vertical asymptotes of $f(x)=\dfrac{x^2+x-2}{x^2-x-6}$.

15. Find the horizontal and vertical asymptotes of $f(x)=\dfrac{x^2-2x-3}{x^2-1}$.

16. Find the horizontal and vertical asymptotes of $f(x)=\dfrac{x^2+5x+4}{x^2+6x+8}$.

17. Find the horizontal and vertical asymptotes of $f(x)=\dfrac{x^2-1}{7x^3-7x^2}$.

18. Find the horizontal and vertical asymptotes of $f(x)=\dfrac{15-12x-3x^2}{50-2x^2}$.

19. Find the horizontal and vertical asymptotes of $f(x)=\dfrac{x^2+x-6}{2x^2-18}$.

20. Sketch the graph of a function that meets all of the following criteria.

1. $f(-1)=3$
2. $\lim_{x\to\infty} f(x)=-2$
3. The line $y=3$ is a horizontal asymptote
4. $\lim_{x\to 2^+} f(x)=-\infty$ and $\lim_{x\to 2^-} f(x)=\infty$
5. $\lim_{x\to -1} f(x)=2$
6. $f(x)$ continuous at every x value except $x=-1$ and $x=2$

21. Sketch the graph of a function that meets all of the following criteria.

1. $f(1)=2$
2. $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to -\infty} f(x)=0$
3. $\lim_{x\to 0^+} f(x)=3$ and $\lim_{x\to 0^-} f(x)=1$
4. Lines $x=2$ and $x=5$ are vertical asymptotes.
5. $\lim_{x\to -4} f(x)=2$
6. f is continuous at all x except $x=0$, $x=2$ and $x=5$

22. Sketch the graph of a function that meets all of the following criteria.

1. The domain of $f(x)$ is all real numbers except $x = 4$
2. $f(x)$ is continuous at all real numbers except $x = 1$ and $x = 4$
3. $\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to 1^+} f(x) = 2$
4. The line $x = 4$ is a vertical asymptote
5. $\lim_{x\to -3} f(x) = 1$

23. Sketch the graph of a function that meets all of the following criteria.

1. $\lim_{x\to 1^+} f(x) = 2$, and $\lim_{x\to 1^-} f(x) = 4$
2. $f(x)$ is continuous at all real numbers except $x = 1$ and $x = 5$
3. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to -\infty} f(x) = 1$
4. The line $x = 5$ is a vertical asymptote

24. Sketch the graph of a function that meets all of the following criteria.

(a) The domain of $f(x)$ is all real numbers except $x = -4$ and $x = 1$

(b) $\lim_{x\to 1^+} f(x) = 3$, and $\lim_{x\to 1^-} f(x) = -1$

(c) $f(x)$ is continuous at all real numbers except $x = -4$ and $x = 1$

(d) $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to -\infty} f(x) = 0$

(e) The line $x = -4$ is a vertical asymptote

25. Sketch the graph of a function that meets all of the following criteria.

1. $f(0) = 2$
2. $f(x)$ is continuous at all real numbers except $x = 1$ and $x = 3$
3. $\lim_{x\to 1^-} f(x) = \infty$ and $\lim_{x\to 1^+} f(x) = -\infty$
4. $\lim_{x\to 3^-} f(x) = 3$ and $\lim_{x\to 3^+} f(x) = 2$
5. $\lim_{x\to -\infty} f(x) = 3$ and $\lim_{x\to\infty} f(x) = 2$

26. Sketch the graph of a function that meets all of the following criteria.

1. The domain of $f(x)$ is all real numbers except $x = -4$ and $x = 1$
2. $\lim_{x\to 1^+} f(x) = 3$, and $\lim_{x\to 1^-} f(x) = -1$
3. $f(x)$ is continuous at all real numbers except $x = -4$ and $x = 1$
4. $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to -\infty} f(x) = 0$
5. The line $x = -4$ is a vertical asymptote

13.4 Exercises Solutions for Chapter 13

1. $\lim_{x\to-\infty} e^{5x-3} = 0$ ($5x-3$ approaches $-\infty$, so e to that power approaches 0)

3. $\lim_{x\to-\infty} \tan^{-1}(x^2) = \frac{\pi}{2}$ (x^2 approaches ∞, so $\tan^{-1}(x^2)$ approaches $\frac{\pi}{2}$)

5. Evaluate the following limits.

(a) $\lim_{x\to 0}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right) = 1+\frac{1}{0-1} = 0$

(b) $\lim_{x\to 1^+}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right) = \infty$

(c) $\lim_{x\to\infty}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right) = 0+0 = 0$

(d) $\lim_{x\to\pi}\left(\frac{\sin(x)}{x}+\frac{1}{x-1}\right) = \frac{\sin\pi}{\pi}+\frac{1}{\pi-1} = \frac{1}{\pi-1}$

7. Answer the following questions about the function $y = f(x)$ graphed below.

(a) $\lim_{x\to\infty} f(x) = -3$

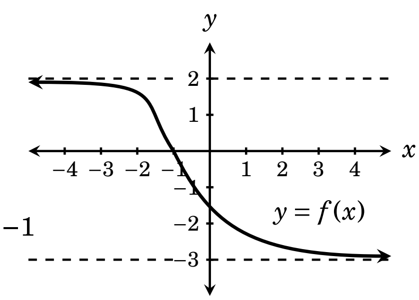
(b) $\lim_{x\to-\infty} f(x) = 2$

(c) $\lim_{x\to-\infty} (f(x)+4) = 2+4 = 6$

(d) $\lim_{x\to\infty} f\left(\frac{1-x}{x}\right) = f\left(\lim_{x\to\infty}\frac{1-x}{x}\right) = f(-1) = 0$

(e) $\lim_{x\to\infty} \cos\big(\pi f(x)\big) = \cos\left(\lim_{x\to\infty}\pi f(x)\right) = \cos(-3\pi) = -1$

(f) $\lim_{x\to\infty} 2^{f(x)} = 2^{-3} = \frac{1}{8}$



9. Find all vertical and horizontal asymptotes of $f(x) = \dfrac{x^2-4}{5x^2-10x}$.

The vertical asymptote of $x = 0$ was found in Chapter 12 Exercise 9. For the horizontal asymptotes, note that $\lim_{x\to\infty}\frac{x^2-4}{5x^2-10x} = \frac{1}{5}$ and $\lim_{x\to-\infty}\frac{x^2-4}{5x^2-10x} = \frac{1}{5}$, so the line $y = \frac{1}{5}$ is the only horizontal asymptote.

11. Find all vertical and horizontal asymptotes of $f(x) = \dfrac{7x^3-7x^2}{x^2-1}$.

The vertical asymptote of $x = -1$ was found in Chapter 12 Exercise 11. For the horizontal asymptotes, note that $\lim_{x\to\infty}\frac{7x^3-7x^2}{x^2-1} = \infty$ and $\lim_{x\to-\infty}\frac{7x^3-7x^2}{x^2-1} = -\infty$, so there is no horizontal asymptote.

13. Find the horizontal and vertical asymptotes of $f(x) = \dfrac{2x^2-8}{x^2+3x+2}$.

The vertical asymptote of $x = -1$ was found in Chapter 12 Exercise 13. For the horizontal asymptotes, note that $\lim_{x\to\infty}\frac{2x^2-8}{x^2+3x+2} = 2$ and $\lim_{x\to-\infty}\frac{2x^2-8}{x^2+3x+2} = 2$, so the line $y = 2$ is the only horizontal asymptote.

15. Find the horizontal and vertical asymptotes of $f(x) = \dfrac{x^2-2x-3}{x^2-1}$.

The vertical asymptote of $x = 1$ was found in Chapter 12 Exercise 15. For the horizontal asymptotes, note that $\lim_{x\to\infty}\frac{x^2-2x-3}{x^2-1} = 1$ and $\lim_{x\to-\infty}\frac{x^2-2x-3}{x^2-1} = 1$, so the line $y = 1$ is the only horizontal asymptote.

17. Find the horizontal and vertical asymptotes of $f(x) = \dfrac{x^2-1}{7x^3-7x^2}$.

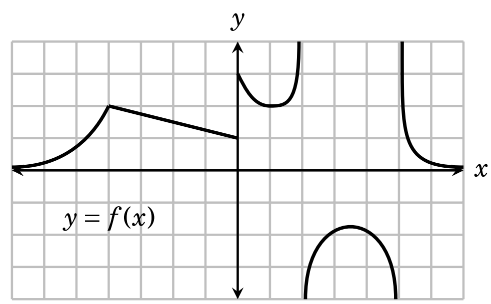
The vertical asymptote of $x = 0$ was found in Chapter 12 Exercise 17. For the horizontal asymptotes, note that $\lim\limits_{x\to\infty} \frac{x^2-1}{7x^3-7x^2} = 0$ and $\lim\limits_{x\to-\infty} \frac{x^2-1}{7x^3-7x^2} = 0$, so the line $y = 0$ is the only horizontal asymptote.

19. Find the horizontal and vertical asymptotes of $f(x) = \dfrac{x^2+x-6}{2x^2-18}$.

The vertical asymptote of $x = 3$ was found in Chapter 12 Exercise 19. For the horizontal asymptotes, note that $\lim\limits_{x\to\infty} \frac{x^2+x-6}{2x^2-18} = \frac{1}{2}$ and $\lim\limits_{x\to-\infty} \frac{x^2+x-6}{2x^2-18} = \frac{1}{2}$, so the line $y = \frac{1}{2}$ is the only horizontal asymptote.

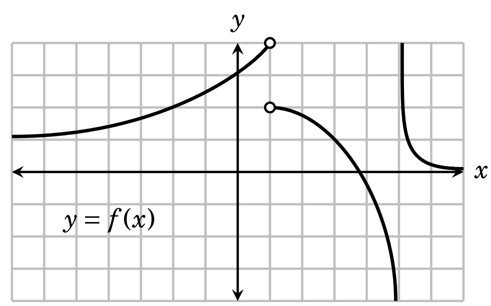
21. Sketch the graph of a function that meets all of the following criteria.

1. $f(1) = 2$
2. $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 0$
3. $\lim\limits_{x\to 0^+} f(x) = 3$ and $\lim\limits_{x\to 0^-} f(x) = 1$
4. x=2 & x=5 are vertical asymptotes.
5. $\lim\limits_{x\to -4} f(x) = 2$
6. f is continuous except at 0, 2 & 5



23. Sketch the graph of a function that meets all of the following criteria.

1. $\lim\limits_{x\to 1^+} f(x) = 2$, and $\lim\limits_{x\to 1^-} f(x) = 4$
2. f continuous except 1 & 5
3. $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 1$
4. line $x = 5$ is a vertical asymptote



25. Sketch the graph of a function that meets all of the following criteria.

1. $f(0) = 2$
2. f continuous except 1 & 3
3. $\lim\limits_{x\to 1^-} f(x) = \infty$ and $\lim\limits_{x\to 1^+} f(x) = -\infty$
4. $\lim\limits_{x\to 3^-} f(x) = 3$ and $\lim\limits_{x\to 3^+} f(x) = 2$
5. $\lim\limits_{x\to-\infty} f(x) = 3$ and $\lim\limits_{x\to\infty} f(x) = 2$