CHAPTER 11

Continuity and Limits of Compositions

The purpose of limits is that they give information about how a function behaves near a "bad point" $x = c$ that is not in its domain. Even if $f(c)$ is not defined, it may be that $\lim_{x \to c} f(x) = L$, for some number L. In this event we know that $f(x)$ becomes ever closer to L as x approaches the forbidden c. Most of our examples in the past several chapters have been of this type.

Of course not every value $x = c$ is a "bad point." It could be that $f(c)$ is defined, and, moreover, $\lim_{x \to c} f(x) = f(c)$. If this is the case for every c in the domain of $f(x)$, then we say that f is *continuous*. Issues concerning whether or not f is continuous are called issues of *continuity*. Exact definitions appear below, but first some general remarks about continuity.

In a first course in calculus it is easy to overlook the huge importance of continuity. And happily, we can (in a first course) almost ignore it. But the theoretical foundation of calculus rests on continuity. In this text and beyond this text are countless theorems having the form

If $\boxed{f \text{ is continuous,}}$ then $\boxed{\text{something significant is true.}}$

Thus continuity is a property that allows us to draw certain important conclusions about a function. If we deal exclusively with continuous functions, then all will be good. From a practical point of view this means that in a first calculus course we need only to understand what continuity is and to recognize which functions possess it. That is this chapter's goal.

11.1 Definitions and Examples

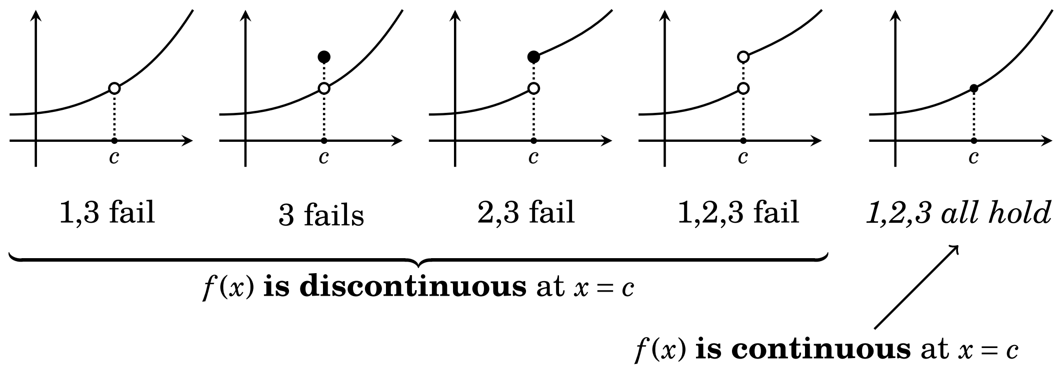
The above discussion motivates our main definition.

Definition 11.1 A function $f(x)$ is **continuous** at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. Note that this means *all* of the following three conditions must be met:

1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists,
3. $\lim_{x \to c} f(x) = f(c)$.

If one or more of these conditions fail, then $f(x)$ is **discontinuous** at c. In such a case we sometimes say that f has a **discontinuity** at c.

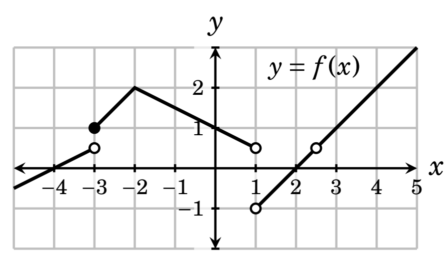
To illustrate this definition, five functions $f(x)$ are graphed below. On the far left, $\lim_{x \to c} f(x)$ exists, but $f(c)$ is not defined. Thus condition 1 fails, so condition 3 also fails by default, so $f(x)$ is not continuous at $x = c$. In the second drawing, $\lim_{x \to c} f(x)$ exists *and* $f(c)$ is defined, *but* $\lim_{x \to c} f(x) \neq f(c)$. Thus condition 3 fails, so $f(x)$ is not continuous at $x = c$.



In the third and fourth drawings $\lim_{x \to c} f(x)$ doesn't exist, so condition 2 fails, so condition 3 also fails (by default) and $f(x)$ is not continuous at c. Only on the far right do all three conditions hold, so $f(x)$ *is* continuous at $x = c$.

Intuitively, $f(x)$ being continuous at $x = c$ means that its graph does not have a "break" at $x = c$. You can trace its graph through $x = c$ without lifting your pencil.

For example, the function on the right is discontinuous at $x = -3$, $x = 1$ and $x = 2.5$. But it is continuous at any other $x = c$ between -5 and 5. You can trace the graph from left to right with a pencil, lifting only when x is $-3, 1$ or 2.5.



Most functions we deal with are continuous at most values of x. For instance, the facts on page 130 state that if $p(x)$ is a polynomial, then $\lim_{x \to c} p(x) = p(c)$ for any number c. According to Definition 11.1, this means any polynomial is continuous at any number $x = c$. This is consistent with our experience with the graphs of polynomials, which are smooth, unbroken curves.

In addition, Chapter 10 showed $\lim_{x \to c} \sin(x) = \sin(c)$ and $\lim_{x \to c} \cos(x) = \cos(c)$ for any number c, meaning $\sin(x)$ and $\cos(x)$ are continuous at any number c. Again, this matches our experience with their graphs, which are continuous unbroken curves. Similarly, our experience with the functions $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$, e^x, b^x, $\ln(x)$ and $\log_b(x)$ suggest that these functions are continuous at any number $x = c$ in their domains.

11.2 Limits of Compositions

One practical application of continuity is that it yields a condition under which we can compute a limit of a composition, like $\lim_{x\to c} f\big(g(x)\big)$. The following theorem gives the conditions under which the limit can brought into the outside function f, as $\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right)$.

> **Theorem 11.1** If $\lim_{x\to c} g(x) = L$ and f is continuous at L, then
> $$\lim_{x\to c} f\big(g(x)\big) \;=\; f\left(\lim_{x\to c} g(x)\right) \;=\; f(L).$$

This formalizes what should be intuitively obvious: If f has no "jump" at L, and $g(x)$ approaches L, then $f\big(g(x)\big)$ will approach $f(L)$. But continuity is essential. This chapter's Exercise 27 asks for an example of an f that is **not** continuous at L and for which $\lim_{x\to c} f\big(g(x)\big) \neq f\left(\lim_{x\to c} g(x)\right)$.

Example 11.1 Find $\lim_{x\to\pi} \cos\left(\dfrac{\pi^2 - x^2}{x-\pi}\right)$.

Because the function cos is continuous at *any* number L, Theorem 11.1 says

$$\begin{aligned}
\lim_{x\to\pi} \cos\left(\frac{\pi^2 - x^2}{x-\pi}\right) &= \cos\left(\lim_{x\to\pi} \frac{\pi^2 - x^2}{x-\pi}\right)\\
&= \cos\left(\lim_{x\to\pi} \frac{(\pi - x)(\pi + x)}{x-\pi}\right)\\
&= \cos\left(\lim_{x\to\pi} -(\pi + x)\right)\\
&= \cos(-2\pi) = 1.
\end{aligned}$$

✍

Example 11.2 Find $\lim_{x\to 1} e^{x^2-2}$.

Because e^x is continuous at *any* number L, Theorem 11.1 guarantees that $\lim_{x\to 1} e^{x^2-2} \;=\; e^{\lim_{x\to 1}(x^2-2)} \;=\; e^{1^2-2} \;=\; e^{-1} \;=\; \dfrac{1}{e}$. ✍

Example 11.3 Find $\lim_{x\to 0} \cos^{-1}\left(\ln\left(\dfrac{\sin(x)}{x}\right)\right)$.

The limit goes first inside the continuous function $\cos^{-1}$ and then inside the continuous function ln.

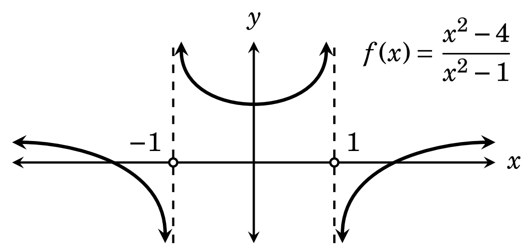
$$\begin{aligned}
\lim_{x\to 0} \cos^{-1}\left(\ln\left(\frac{\sin(x)}{x}\right)\right) &= \cos^{-1}\left(\lim_{x\to 0} \ln\left(\frac{\sin(x)}{x}\right)\right)\\
&= \cos^{-1}\left(\ln\left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)\right) = \cos^{-1}\big(\ln(1)\big) = \cos^{-1}(0) = \frac{\pi}{2}.
\end{aligned}$$

✍

11.3 Continuity on Intervals

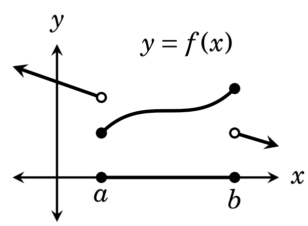
Typically a function will be continuous at most points $x = c$. Discontinuities are anomalies. The function $\sin(x)$, for example, is continuous at every number $x = c$ in its domain $(-\infty, \infty)$. In fact, the vast majority of the functions we deal with routinely are continuous on their domains.

Take the function f on the right. It is discontinuous at $x = 1$ and $x = -1$, which are not in its domain. At any other number $x = c$ we have $\lim_{x \to c} f(x) = f(c)$, so f is continuous at c. As f is continuous at every x except ± 1, we say f is continuous on the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.



In general we say a function f is continuous on an open interval (a, b) if it is continuous at every c in (a, b), that is, if $\lim_{x \to c} f(x) = f(c)$ when $a < c < b$. Informally this means that f has no "jumps" on (a, b).

Now think about what it means for a function to be continuous on a *closed* interval $[a, b]$. Intuitively this means that it has no "jumps" on $[a, b]$. Thus we would consider the function on the right to be continuous on $[a, b]$. Even though it is discontinuous at the endpoints a and b, the discontinuities disappear if we erase the parts of the graph outside of $[a, b]$.



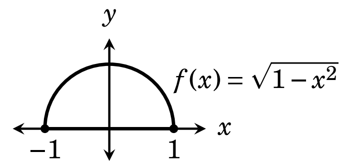
We can formulate this precisely with right- and left-hand limits. Saying that f is continuous on the closed interval $[a, b]$ means that for any number c in $[a, b]$ we have $\lim_{x \to c} f(x) = f(c)$, *as x remains in $[a, b]$ as it approaches c*. But if x is in $[a, b]$, then it can approach a only from the right, and b only from the left. Thus we require $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

Let's record these ideas of this page in a definition.

Definition 11.2 (Continuity on intervals)

- A function f is continuous on (a, b) if $\lim_{x \to c} f(x) = f(c)$ when $a < c < b$.
- f is continuous on $[a, b]$ if it is continuous on (a, b), and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.
- f is continuous on $[a, b)$ if it is continuous on (a, b), and $\lim_{x \to a^+} f(x) = f(a)$.
- f is continuous on $(a, b]$ if it is continuous on (a, b), and $\lim_{x \to b^-} f(x) = f(b)$.

As an example, consider the function $f(x) = \sqrt{1-x^2}$. Its graph is the upper half of the unit circle. According to Definition 11.2 this function is continuous on the closed interval $[-1,1]$, as follows.

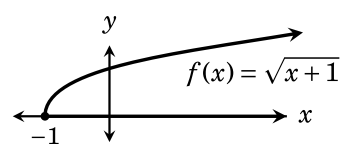


First let's check that it is continuous on the open interval $(-1,1)$. If c is in this interval then limit laws give

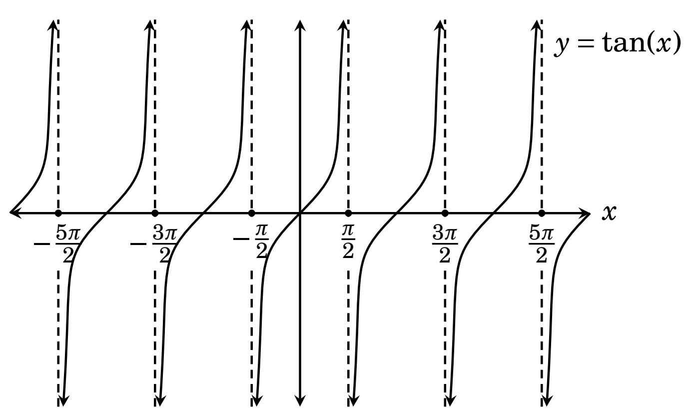
$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{1-x^2} = \sqrt{1-c^2} = f(c).$$

That is, $\lim_{x\to c} f(x) = f(c)$, so f is continuous on $(-1,1)$. Concerning the endpoints, neither $\lim_{x\to -1} f(x)$ nor $\lim_{x\to 1} f(x)$ exist because $f(x)$ is undefined when x is to the left of -1 or to the right of 1. But we do have $\lim_{x\to -1^+} f(x) = 0 = f(-1)$ and $\lim_{x\to 1^-} f(x) = 0 = f(1)$. Thus $\sqrt{1-x^2}$ is continuous on $[-1,1]$.

Our next example is $f(x) = \sqrt{x+1}$, whose graph is the graph of $y = \sqrt{x}$ shifted one unit left. Note that $f(x)$ is continuous on its domain $[-1,\infty)$ because it is continuous on $(-1,\infty)$, and $\lim_{x\to -1^+} f(x) = 0 = f(-1)$.



Our final example concerns the function $\tan(x)$. This function has infinitely many discontinuities, at $\frac{\pi}{2} + k\pi$ for any integer k. But if c is not one of these numbers, then $\lim_{x\to c} \tan(x) = \tan(c)$. Thus $\tan(x)$ is continuous on each of the intervals $\left(-\frac{\pi}{2} + k\pi,\ \frac{\pi}{2} + k\pi\right)$.



In general the domain of a function is an interval or a collection of intervals. Each example on this page features a function that is continuous on its domain. In fact, almost all of the functions we deal with in calculus are continuous on their domains, as the next section explains.

11.4 Building Continuous Functions

In using calculus it is often important that the functions we deal with are continuous. Fortunately most are. The next theorem gives a list of basic functions that are continuous on their domains. (In this theorem k is a constant real number, and is interpreted as the constant function $f(x) = k$ whose graph is a horizontal line crossing the y-axis at k. Similarly x represents the identity function $f(x) = x$ whose graph is a straight line with slope 1 and y-intercept 0. Also a is a positive constant.)

Theorem 11.2 Basic Continuous Functions
The following functions are continuous on their domains:

k	x	$\lvert x\rvert$	a^x	$\ln(x)$	$\log_a(x)$
$\sin(x)$	$\cos(x)$	$\tan(x)$	$\csc(x)$	$\sec(x)$	$\cot(x)$
$\sin^{-1}(x)$	$\cos^{-1}(x)$	$\tan^{-1}(x)$	$\csc^{-1}(x)$	$\sec^{-1}(x)$	$\cot^{-1}(x)$

Two continuous functions $f(x)$ and $g(x)$ can be combined by various algebraic operations and the result is continuous. For example, if they are both continuous at c, then their product $f(x)\cdot g(x)$ is continuous at c because a limit law gives

$$\lim_{x\to c} f(x)\cdot g(x) = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right) = f(c)g(c).$$

Here is a summary of ways that continuous functions can be combined to yield new continuous functions.

Theorem 11.3 Building Continuous Functions
If $f(x)$ and $g(x)$ are continuous on their domains, then so are the following.

$f(x)+g(x)$	$f(x)-g(x)$	$k\cdot f(x)$
$f(x)\cdot g(x)$	$\dfrac{f(x)}{g(x)}$	$\lvert f(x)\rvert$
$f\big(g(x)\big)$	$\big(f(x)\big)^n$	$\sqrt[n]{f(x)}$

The main point of this theorem is that if a function is built up by combining continuous functions with the stated operations, then it itself is continuous. For example,

$$h(x) = \frac{\cos(x)+x^2}{\sin(x)} + 5\sqrt{x}$$

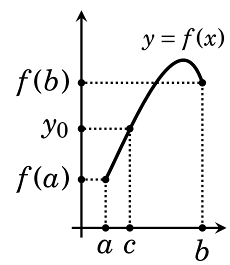
is continuous on its domain because it's built up by combining the continuous functions x, $\sin(x)$ and $\cos(x)$ with operations listed above.

11.5 The Intermediate Value Theorem

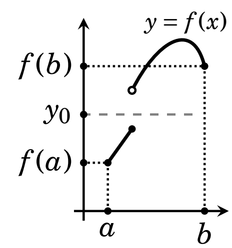
Though the theorem we now discuss is not the most important result in a calculus course, it is a good example of a theorem having form "*If $f(x)$ is continuous, then something significant is true*," promised on page 156.

Theorem 11.4 Intermediate Value Theorem
If $f(x)$ is continuous on a closed interval $[a,b]$, and y_0 is any number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ for which $f(c) = y_0$.

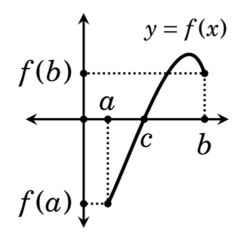
That is, a continuous function on $[a,b]$, starting at height $f(a)$ and ending at height $f(b)$, attains every height y_0 between $f(a)$ and $f(b)$.



The intermediate value theorem says something very intuitive about a function that is continuous on $[a,b]$, namely that it must take on every value between $f(a)$ and $f(b)$. Notice that continuity is an essential ingredient. If $f(x)$ were not continuous, then there might be heights y_0 between $f(a)$ and $f(b)$ that don't equal any $f(c)$, as shown on the right.



One application of Theorem 11.4 is to equations of form $f(x) = 0$. If there are numbers a and b for which one of $f(a)$ and $f(b)$ is positive and the other is negative, and $f(x)$ is continuous on $[a,b]$, then we know the equation $f(x) = 0$ has a solution c in $[a,b]$. This is because $y_0 = 0$ is between $f(a)$ and $f(b)$, so Theorem 11.4 guarantees a c in $[a,b]$ with $f(c) = 0$.



Example 11.4 Show that the equation $\cos(x) = 2x$ has at least one solution.

This equation can't be solved with standard algebraic techniques because x cannot be isolated. (And writing it as $\cos(x) - 2x = 0$, we notice that it is impossible to factor.) This problem is asking us just to show that there exists a solution, not what number that solution is. To answer the question, notice that the function $f(x) = \cos(x) - 2x$ is continuous because it is built from continuous functions $\cos(x)$ and x by operations listed in Theorem 11.3. Notice that $f(0) = \cos(0) + 2 \cdot 0 = 1$ is positive but $f(\pi) = \cos(\pi) - 2\pi = -1 - 2\pi$ is negative, so the number 0 is between $f(0)$ and $f(\pi)$. The intermediate value theorem guarantees a number c in $[0,\pi]$ for which $f(c) = 0$. This means $\cos(c) - 2c = 0$, so c is a solution to $\cos(x) = 2x$. ✍

Exercises for Chapter 11

1. Find: $\lim_{x\to\pi/2} \ln\big(\sin(x)\big)$

2. Find: $\lim_{x\to\sqrt{3}} \tan^{-1}(x)$

3. Find: $\lim_{x\to 0} \tan^{-1}\left(\frac{\sin(x)}{x}\right)$

4. Find: $\lim_{x\to\pi/2} e^{\cos(x)}$

5. Find: $\lim_{x\to\pi/2} 2^{3\cos(2x)}$

6. Find: $\lim_{x\to 1} \ln\left(\frac{x^2-1}{2x-2}\right)$

7. Find: $\lim_{x\to\pi} \cos\left(\frac{x}{3}\right)$

8. Find: $\lim_{x\to -1^+} \sin^{-1}(x)$

9. Find: $\lim_{x\to 0} \sin\left(\frac{\pi x + x^2}{4x}\right)$

10. Find: $\lim_{x\to 4} \log_2\left(\frac{x^2-16}{x-4}\right)$

11. State the intervals on which the function $y = \frac{x+1}{x^2-4x+3}$ is continuous.

12. State the intervals on which the function $y = \frac{\sqrt{x+5}}{e^x - 1}$ is continuous.

13. State the intervals on which the function $y = \sqrt{x^2-5}$ is continuous.

14. State the intervals on which the function $y = \frac{\sin(x)}{x}$ is continuous.

15. Draw the graph of a function that meets *all five* of the following conditions.

1. $f(x)$ is continuous everywhere except at $x = 1$ and $x = 2$.
2. $f(3) = 1$
3. $\lim_{x\to 1} f(x) = -1$
4. $\lim_{x\to 2^-} f(x) = 1$
5. $\lim_{x\to 2^+} f(x) = 2$

16. Draw the graph of a function that meets *all five* of the following conditions.

1. $f(x)$ is continuous everywhere except at $x = -1$ and $x = 1$.
2. $f(3) = 2$
3. $\lim_{x\to -1} f(x) = 2$
4. $\lim_{x\to 1^-} f(x) = 1$
5. $\lim_{x\to 1^+} f(x) = -1$

17. Find the value a such that f is continuous on $(-\infty,\infty)$:

$$f(x) = \begin{cases} 3x-2 & \text{if } x < 2 \\ 5x+a & \text{if } x \ge 2 \end{cases}$$

18. Find the value a such that f is continuous on $(-\infty,\infty)$:

$$f(x) = \begin{cases} x^2-2 & \text{if } x < 3 \\ ax & \text{if } x \ge 3 \end{cases}$$

19. Find the value a such that f is continuous on $(-\infty,\infty)$:

$$f(x) = \begin{cases} x^2+2 & \text{if } x < 3 \\ ax & \text{if } x \ge 3 \end{cases}$$

20. Find the value a such that f is continuous on $(-\infty,\infty)$:

$$f(x) = \begin{cases} \dfrac{\sin(3x-3)}{x-1} & \text{if } x \ne 1 \\ a & \text{if } x = 1 \end{cases}$$

21. Answer the questions about the function $f(x)$ graphed below.

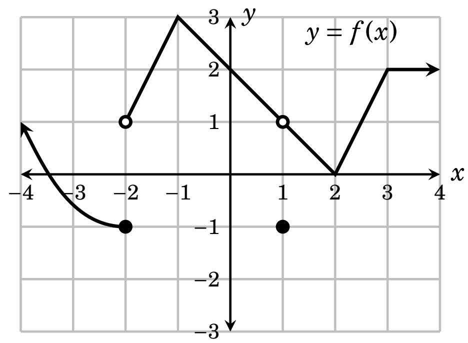
(a) At which values c is $f(x)$ **not** continuous at $x = c$?

(b) $f\big(f(1)\big) =$

(c) $\lim_{x \to 1} f\big(f(x)\big) =$

(d) $f\big(f(-1)\big) =$

(e) $\lim_{x \to -1} f\big(f(x)\big) =$



22. Answer the questions about the function $f(x)$ graphed below.

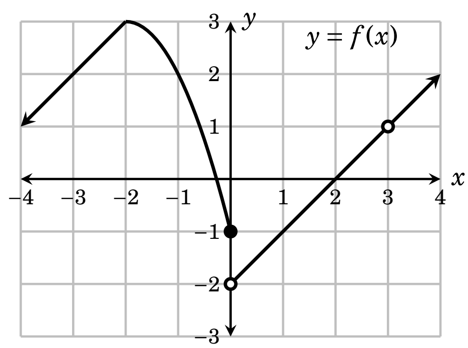
(a) At which values c is $f(x)$ **not** continuous at $x = c$?

(b) $\lim_{x \to 2} f\left(\dfrac{x^2-4}{x-2}\right) =$

(c) $\lim_{x \to -1} \dfrac{\big(f(x)\big)^2 - 4}{f(x) - 2} =$

(d) $\lim_{x \to 3} f \circ f(x) =$

(e) $\lim_{x \to 3} \dfrac{5f(x)}{1 + f(x)} =$



23. Answer these questions about the functions f and g graphed below.

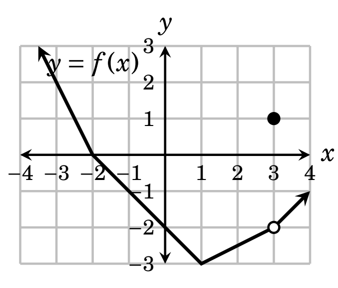
(a) $f(3) =$

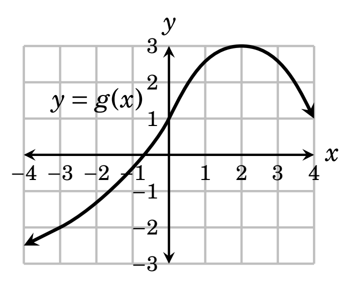
(b) $\lim_{x \to 2} g(x) =$

(c) $f\left(\lim_{x \to 2} g(x)\right) =$

(d) $\lim_{x \to 2} f\big(g(x)\big) =$

(e) $\lim_{x \to 3} \dfrac{f(x)}{g(x-1)} =$





24. Answer these questions about the functions f and g graphed below.

(a) $\lim_{x \to 1} f(x)g(x) =$

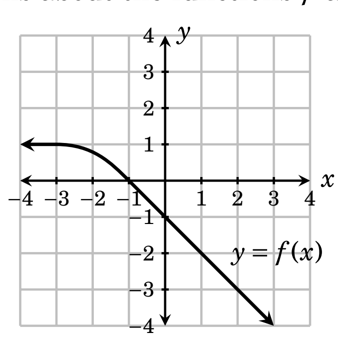
(b) $\lim_{x \to 1} f\big(g(x)\big) =$

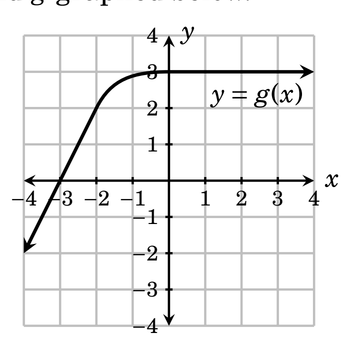
(c) $\lim_{x \to -2} f\big(g(x)\big) =$

(d) $\lim_{x \to 2} g\big(f(x)\big) =$

(e) $\lim_{x \to -1} g\big(f(-2x)\big) =$

(f) $\lim_{x \to 2} f\big(f(x)\big) =$





25. Answer these questions about the functions f and g graphed below.

(a) $\lim_{x\to 1} f(x)g(x) =$

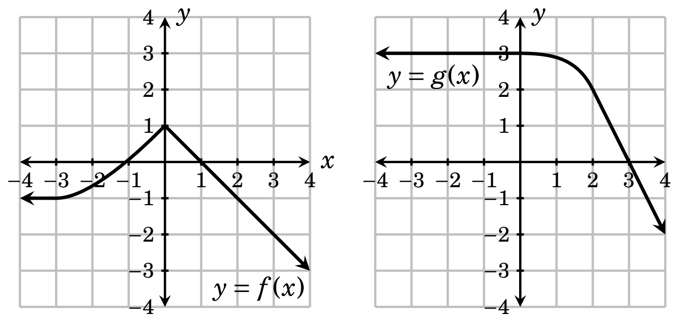
(b) $\lim_{x\to 0} f\big(g(x)\big) =$

(c) $\lim_{x\to 2} f\big(g(x)\big) =$

(d) $\lim_{x\to 2} g\big(f(x)\big) =$

(e) $\lim_{x\to -1} g\big(f(x)\big) =$

(f) $\lim_{x\to 2} f\big(f(x)\big) =$



26. Answer these questions about the functions f and g graphed below.

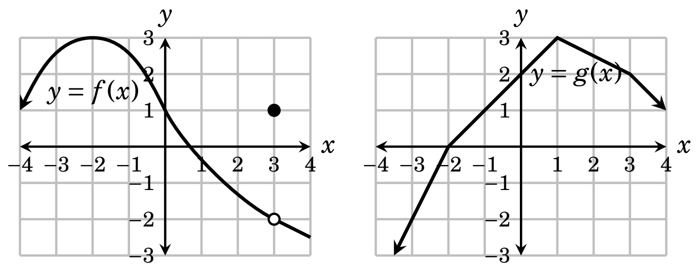
(a) $\lim_{x\to 1} g(x) =$

(b) $f(3) =$

(c) $f\left(\lim_{x\to 1} g(x)\right) =$

(d) $\lim_{x\to 1} f\big(g(x)\big) =$

(e) $\lim_{x\to 3} f(x)g(x) =$



27. Show that Theorem 11.1 requires continuity: Find functions f and g for which $\lim_{x\to c} g(x)=L$, f is **not** continuous at L, and $\lim_{x\to c} f\big(g(x)\big) \neq f\left(\lim_{x\to c} g(x)\right)$. Hint: you will find several such examples in the exercises above.

28. Use the intermediate theorem to show that the equation $x^3 + x + \sin(x) = 11$ has a solution.

29. Use the intermediate theorem to show that the equation $e^x = 7 - x$ has a solution.

11.6 Exercise Solutions for Chapter 11

1. $\lim_{x\to\pi/2} \ln\big(\sin(x)\big) = \ln\left(\lim_{x\to\pi/2}\sin(x)\right) = \ln(1) = 0$

3. $\lim_{x\to 0}\tan^{-1}\left(\frac{\sin(x)}{x}\right) = \tan^{-1}\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

5. $\lim_{x\to\pi/2} 2^{3\cos(2x)} = 2^{3\cos(2\cdot\pi/2)} = 2^{3\cos(\pi)} = 2^{-3} = \frac{1}{8}$

7. $\lim_{x\to\pi}\cos\left(\frac{x}{3}\right) = \cos\left(\lim_{x\to\pi}\frac{x}{3}\right) = \cos(\pi/3) = \frac{1}{2}$

9. $\lim_{x\to 0}\sin\left(\frac{\pi x + x^2}{4x}\right) = \sin\left(\lim_{x\to 0}\frac{\pi x + x^2}{4x}\right) = \sin\left(\lim_{x\to 0}\frac{x(\pi + x)}{4x}\right) = \sin\left(\lim_{x\to 0}\frac{\pi + x}{4}\right) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$

11. State the intervals on which the function $y = \frac{x+1}{x^2-4x+3}$ is continuous.

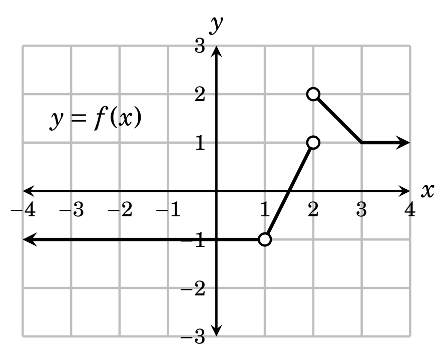
This is a rational function, so it will be continuous on its domain. Given that $y = \frac{x+1}{(x-1)(x-3)}$, its domain is all real numbers except 1 and 3. Therefore this function is continuous on $(-\infty,1)\cup(1,3)\cup(3,\infty)$.

13. State the intervals on which the function $y = \sqrt{x^2-5}$ is continuous.

By Theorem 11.3, this function is continuous on its domain, which is $(-\infty,-\sqrt{5}]\cup[\sqrt{5},\infty)$.

15. Draw the graph of a function that meets *all five* of the following conditions.

1. $f(x)$ is continuous everywhere except at $x=1$ and $x=2$.
2. $f(3)=1$
3. $\lim_{x\to 1} f(x) = -1$
4. $\lim_{x\to 2^-} f(x) = 1$
5. $\lim_{x\to 2^+} f(x) = 2$



17. Find the value a such that f is continuous on $(-\infty,\infty)$: $f(x) = \begin{cases} 3x-2 & \text{if } x<2 \\ 5x+a & \text{if } x\geq 2 \end{cases}$

This function is a polynomial $f(x)=3x-2$ on $(-\infty,2)$, so it is continuous on that interval. The function is $f(x)=5x+a$ on $(2,\infty)$, so it is continuous on that interval. Thus the only possible location for a discontinuity is at $x=2$. In order for f to be continuous at $x=2$, we must have $\lim_{x\to 2} f(x) = f(2)$. Now, $f(2) = 5\cdot 2 + a = 10+a$, so we require $\lim_{x\to 2} f(x) = 10+a$. In particular, $10+a = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(3x-2) = 4$. This gives $a=-6$.

19. Find the value a such that f is continuous on $(-\infty,\infty)$: $f(x) = \begin{cases} x^2+2 & \text{if } x<3 \\ ax & \text{if } x \geq 3 \end{cases}$

This function is a polynomial $f(x) = x^2+2$ on $(-\infty, 3)$, so it is continuous on that interval. The function is $f(x) = ax$ on $(3,\infty)$, so it is continuous on that interval. Thus the only possible location for a discontinuity is at $x = 3$. In order for f to be continuous at $x = 3$, we must have $\lim_{x\to 3} f(x) = f(3)$. Now, $f(3) = a \cdot 3$, so we require $\lim_{x\to 3} f(x) = 3a$. In particular, $3a = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(x^2+2) = 11$. This gives $a = \frac{11}{3}$.

21. Answer the questions about the function $f(x)$ graphed below.

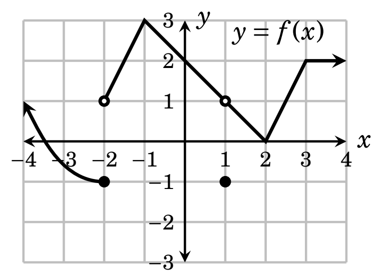
(a) At which values c is $f(x)$ **not** continuous at x=c?
Answer: −2 and 1

(b) $f\big(f(1)\big) = f(-1) = 3$

(c) $\lim_{x\to 1} f\big(f(x)\big) = 1$

(d) $f\big(f(-1)\big) = f(3) = 2$

(e) $\lim_{x\to -1} f\big(f(x)\big) = 2$



23. Answer these questions about the functions f and g graphed below.

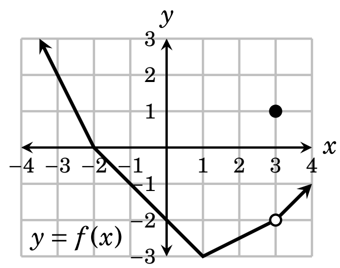
(a) $f(3) = 1$

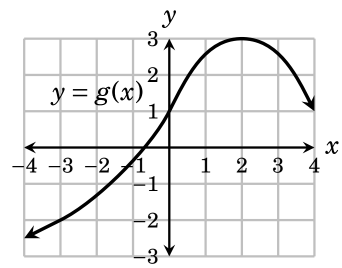
(b) $\lim_{x\to 2} g(x) = 3$

(c) $f\left(\lim_{x\to 2} g(x)\right) = f(3) = 1$

(d) $\lim_{x\to 2} f\big(g(x)\big) = -2$

(e) $\lim_{x\to 3} \dfrac{f(x)}{g(x-1)} = \dfrac{-2}{3}$





25. Answer these questions about the functions f and g graphed below.

(a) $\lim_{x\to 1} f(x)g(x) = 0$

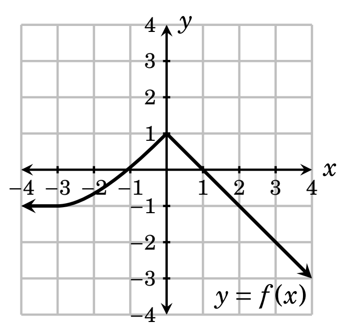
(b) $\lim_{x\to 0} f\big(g(x)\big) = -2$

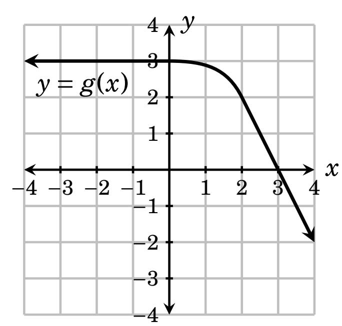
(c) $\lim_{x\to 2} f\big(g(x)\big) = -1$

(d) $\lim_{x\to 2} g\big(f(x)\big) = 3$

(e) $\lim_{x\to -1} g\big(f(x)\big) = 3$

(f) $\lim_{x\to 2} f\big(f(x)\big) = 0$





27. Show that Theorem 11.1 requires continuity: Find functions f and g for which $\lim_{x\to c} g(x)$=L, f is **not** continuous at L, and $\lim_{x\to c} f\big(g(x)\big) \neq f\left(\lim_{x\to c} g(x)\right)$.

Answer: In Exercise 23 above, we saw functions f and g for which $\lim_{x\to 2} g(x)$=3, f is **not** continuous at 3, and $\lim_{x\to 2} f\big(g(x)\big) \neq f\left(\lim_{x\to 2} g(x)\right)$.

29. Use the intermediate theorem to show that the equation $e^x = 7 - x$ has a solution.

This amounts to showing that $e^x + x - 7 = 0$ has a solution. Let $f(x) = e^x + x - 7$, which is continuous on $(-\infty, \infty)$. We need to show that $f(x) = 0$ has a solution. Notice that $f(0) = e^0 + 0 - 7 = -6$ is negative but $f(7) = e^7 + 7 - 7 = e^7$ is positive. Since f is continuous on $[0,7]$ and $f(0) < 0 < f(7)$, the intermediate value theorem guarantees a number $0 < c < 7$ for which $f(c) = 0$. Therefore c is a solution for $e^x + x - 7 = 0$.