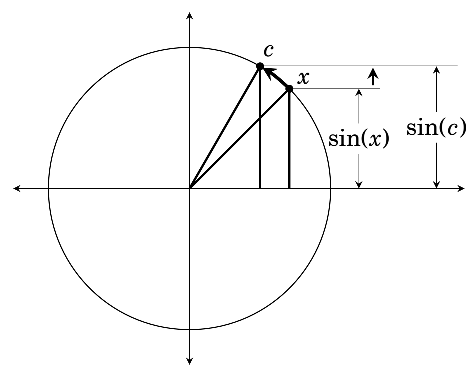
CHAPTER 10

Limits of Trigonometric Functions

Some limits involve trigonometric functions. This Chapter explains how to deal with them. Let's begin with the six trigonometric functions.

10.1 Limits of the Six Trigonometric Functions

We start with the simple limit $\lim\limits_{x \to c} \sin(x)$. Here x is a radian measure because we are taking sin of it. And because the radian measure x approaches c, we interpret c as a radian measure too. The picture on the right illustrates this. The point x on the unit circle moves toward the point c on the circle. As this happens, $\sin(x)$ approaches the number $\sin(c)$. Thus $\lim\limits_{x \to c} \sin(x) = \sin(c)$.



For example, $\lim\limits_{x \to \frac{\pi}{4}} \sin(x) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. With a slight adaption, the above picture also shows $\lim\limits_{x \to c} \cos(x) = \cos(c)$. And applying limit law 5, we get

$$\lim_{x \to c} \tan(x) \;=\; \lim_{x \to c} \frac{\sin(x)}{\cos(x)} \;=\; \frac{\lim\limits_{x \to c} \sin(x)}{\lim\limits_{x \to c} \cos(x)} \;=\; \frac{\sin(c)}{\cos(c)} \;=\; \tan(c),$$

provided that $\cos(c) \neq 0$, that is, $c \neq \frac{\pi}{2} + k\pi$, where k is an integer. In this way that we get the following formulas.

$\lim\limits_{x \to c} \sin(x) = \sin(c)$	for all real numbers c
$\lim\limits_{x \to c} \cos(x) = \cos(c)$	for all real numbers c
$\lim\limits_{x \to c} \tan(x) = \tan(c)$	for all real numbers $c \neq \frac{\pi}{2} + k\pi$
$\lim\limits_{x \to c} \sec(x) = \sec(c)$	for all real numbers $c \neq \frac{\pi}{2} + k\pi$
$\lim\limits_{x \to c} \cot(x) = \cot(c)$	for all real numbers $c \neq k\pi$
$\lim\limits_{x \to c} \csc(x) = \csc(c)$	for all real numbers $c \neq k\pi$

Example 10.1 Find $\lim\limits_{x\to\pi} \dfrac{\cos(x)}{x^2}$.

Because the denominator does not approach zero, we can use limit law 5 with the rules just derived. Then $\lim\limits_{x\to\pi} \dfrac{\cos(x)}{x^2} = \dfrac{\lim\limits_{x\to\pi}\cos(x)}{\lim\limits_{x\to\pi} x^2} = \dfrac{\cos(\pi)}{\pi^2} = \dfrac{-1}{\pi^2}$. ✍

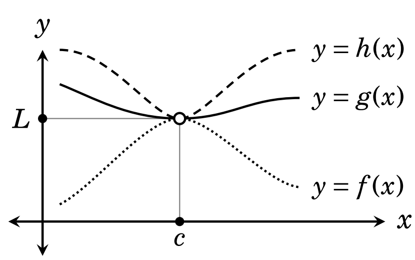
Example 10.2 Find $\lim\limits_{x\to\pi/4} \dfrac{8x\tan(x) - 2\pi\tan(x)}{4x-\pi}$.

Here the denominator approaches zero, so we try to factor and cancel: $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{8x\tan(x)-2\pi\tan(x)}{4x-\pi} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{2\tan(x)\big(4x-\pi\big)}{4x-\pi} = \lim\limits_{x\to\frac{\pi}{4}} 2\tan(x) = 2\tan\left(\dfrac{\pi}{4}\right) = 2.$ ✍

10.2 The Squeeze Theorem and Two Important Limits

It is easy to imagine limits where factoring and canceling is impossible, or for which the limit laws do not apply. For example, in $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$ we can't factor an x from the top to cancel the x on the bottom (which approaches 0). Actually, this particular limit turns out to be significant in calculus. We now discuss a theorem that handles limits such as this one. The idea is to cleverly compare a complicated limit to two simpler limits.

Theorem 10.1 (The Squeeze Theorem)
Suppose we need to compute $\lim\limits_{x\to c} g(x)$. Suppose also that we can find two functions $f(x)$ and $h(x)$ for which $f(x) \le g(x) \le h(x)$ for values of x near c, and for which $\lim\limits_{x\to c} f(x) = L = \lim\limits_{x\to c} h(x)$. Then $\lim\limits_{x\to c} g(x) = L$.

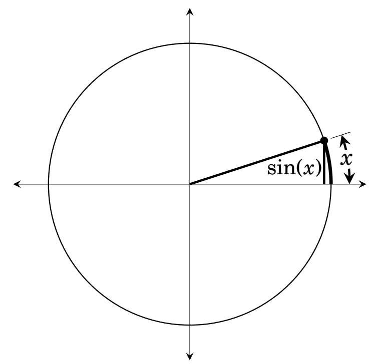


The above picture illustrates the squeeze theorem. The graph of $g(x)$ is squeezed between the graphs of $f(x)$ and $h(x)$, both of which approach L as x approaches c. The squeeze theorem states the obvious fact that in this situation we can conclude that $g(x)$ approaches L too.

Applying the squeeze theorem to find $\lim\limits_{x\to c} g(x)$ requires some ingenuity. We have to find two other functions $f(x)$ and $h(x)$ for which $f(x) \le g(x) \le h(x)$ **and** both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} h(x)$ are easy to compute **and** are both equal to the same number L. At that point the squeeze theorem says $\lim\limits_{x\to c} g(x) = L$.

We will next use the squeeze theorem to find $\lim_{x\to 0} \frac{\sin(x)}{x}$, which will be needed in Chapter 21. But first let's think about what we'd expect it to be.

The unit circle on the right shows a radian measure x, close to 0. the vertical side of the triangle is the corresponding value $\sin(x)$. Both $\sin(x)$ and x are small, but the curved arc x is so small that it looks almost like a vertical line. The smaller x, the more "vertical" it looks, and in fact it becomes almost indistinguishable from the vertical side $\sin(x)$. For very small x the ratio $\frac{\sin(x)}{x}$ appears to be quite close to 1. We might guess $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

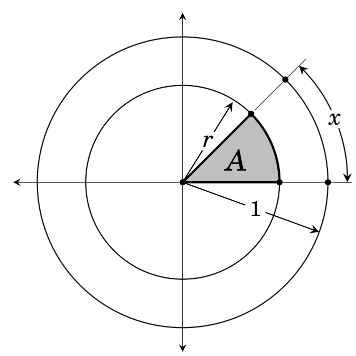


In fact, this turns out to be exactly the case. Proving it with the squeeze theorem requires a formula from geometry. Recall that a **sector of a circle** is a "pie slice" of the circle, as illustrated below, shaded.

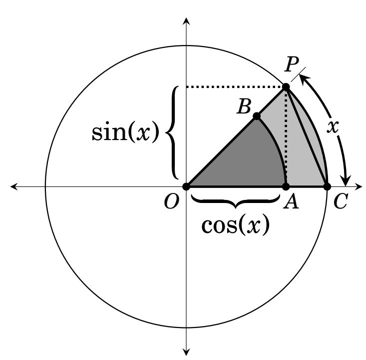
Formula: The area of a sector of a circle of angle x and radius r is

$$A = \frac{1}{2}r^2 x.$$

Here is why the formula works: The area of a circle of radius r is πr^2. The sector takes up only a fraction of this circle, that fraction being x radians out of 2π radians around the entire unit circle, or $\frac{x}{2\pi}$. Thus $A = \pi r^2 \cdot \frac{x}{2\pi} = \frac{1}{2}r^2 x$.



We are ready to carry out our plan of proving that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ via the squeeze theorem. We will concoct two functions $f(x)$ and $h(x)$ with $f(x) \le \frac{\sin(x)}{x} \le h(x)$ and $\lim_{x\to 0} f(x) = 1 = \lim_{x\to 0} h(x)$. The functions f and h will come from the diagram on the right showing a sector OCP on the unit circle, and another sector OAB of radius $\cos(x)$ inside it. From this we get the following:



$$\begin{pmatrix}\text{Area of}\\ \text{sector } OAB\end{pmatrix} \le \begin{pmatrix}\text{Area of}\\ \text{triangle } OCP\end{pmatrix} \le \begin{pmatrix}\text{Area of}\\ \text{sector } OCP\end{pmatrix}.$$

Using the area formula for a sector (from the previous page) and the area formula for a triangle (from heart), this becomes

$$\frac{1}{2}\cdot\cos^2(x)\cdot x \;\le\; \frac{1}{2}\cdot 1\cdot\sin(x) \;\le\; \frac{1}{2}\cdot 1^2\cdot x.$$

Actually, this only works if x is *positive*. If it were negative, then the above "areas" would be negative too. We correct this by taking the absolute value of the potentially negative terms x and $\sin(x)$.

$$\frac{1}{2}\cdot\cos^2(x)\cdot|x| \;\le\; \frac{1}{2}\cdot 1\cdot|\sin(x)| \;\le\; \frac{1}{2}\cdot 1^2\cdot|x|.$$

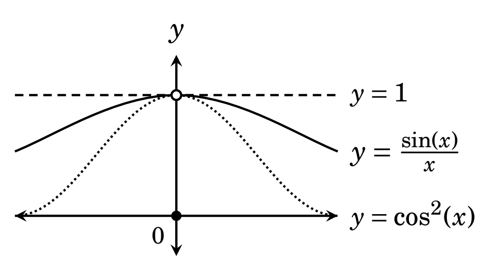
Now multiply all parts of this inequality by the positive number $\frac{2}{|x|}$ to get

$$\cos^2(x) \;\le\; \frac{|\sin(x)|}{|x|} \;\le\; 1.$$

At this point the absolute values are unnecessary because if x is close to zero (as it is when $x \to 0$), then x and $\sin(x)$ are either both positive or both negative, so $\frac{\sin(x)}{x}$ is already positive. Updating the above, we get

$$\cos^2(x) \;\le\; \frac{\sin(x)}{x} \;\le\; 1.$$

Now we've squeezed $y = \frac{\sin(x)}{x}$ between the functions $y = \cos^2(x)$ and $y = 1$.

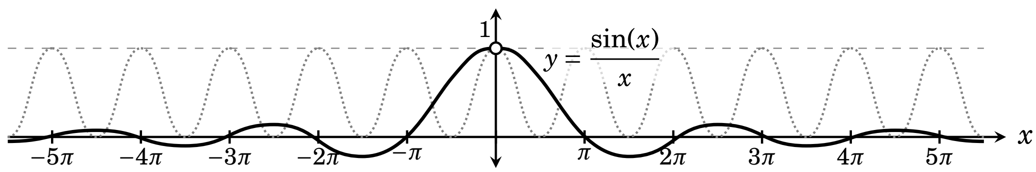


Because $\lim_{x\to 0}\cos^2(x) = \cos^2(0) = 1 = \lim_{x\to 0} 1$, the squeeze theorem guarantees

$$\boxed{\lim_{x\to 0}\frac{\sin(x)}{x} = 1.} \tag{10.1}$$

From this day forward, remember the fundamental fact $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$.

Below is a more complete picture of this situation, showing $y = \frac{\sin(x)}{x}$ with $y = \cos^2(x)$ and $y = 1$. Notice that it's not the case that $\cos^2(x) \leq \frac{\sin(x)}{x} \leq 1$ for *every* value of x. But this *does* hold when x is near zero, and that is all we needed to apply the squeeze theorem.



Students often assert incorrectly that $\frac{\sin(x)}{x} = 1$. But that plainly wrong. The above graph shows that $\frac{\sin(x)}{x}$ **never** equals 1. In fact, $\frac{\sin(x)}{x} < 1$ for any x except 0, and it is undefined when $x = 0$. What we have determined is that it grows ever closer to 1 as x approaches zero, that is, $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$.

Now we use this fact to compute another significant limit.

Example 10.3 Find $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x}$.

Of course we can't just plug in $x = 0$ because that would give the $\frac{0}{0}$ nonsense. Nor can we factor anything from the top to cancel with the x on the bottom. But let's entertain a little wishful thinking. If we could only change the top from $\cos(x)-1$ to $\cos^2(x)-1$, then the identity $\sin^2(x)+\cos^2(x) = 1$ would turn the top into $\cos^2(x)-1 = -\sin(x)$, and we'd get our familiar form $\frac{\sin(x)}{x}$. We can accomplish just this by multiplying by the conjugate of $\cos(x)-1$.

$$
\begin{aligned}
\lim_{x\to 0} \frac{\cos(x)-1}{x} &= \lim_{x\to 0} \frac{\cos(x)-1}{x} \cdot \frac{\cos(x)+1}{\cos(x)+1} && \longleftarrow \text{ multiply by } 1 = \frac{\cos(x)+1}{\cos(x)+1} \\
&= \lim_{x\to 0} \frac{\cos^2(x)-1}{x(\cos(x)+1)} && \longleftarrow \text{ FOIL top} \\
&= \lim_{x\to 0} \frac{-\sin^2(x)}{x(\cos(x)+1)} && \longleftarrow \text{ use } \cos^2(x)-1 = -\sin^2(x) \\
&= \lim_{x\to 1} \frac{\sin(x)}{x} \cdot \frac{-\sin(x)}{\cos(x)+1} && \longleftarrow \text{ regroup} \\
&= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{-\sin(x)}{\cos(x)+1} && \longleftarrow \text{ apply limit laws} \\
&= 1 \cdot \frac{\sin(0)}{\cos(0)+1} && \longleftarrow \text{ use } \lim_{x\to 0} \frac{\sin(x)}{x} = 1, \text{ limit laws} \\
&= 1 \cdot \frac{0}{1+1} \; = \; 0 && \longleftarrow \text{ final answer!}
\end{aligned}
$$

Therefore $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x} = 0$. ✍

Here is a summary of what we developed over the previous three pages. These limits will be useful later, and should be remembered.

Theorem 10.2 (Two Important Limits)

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \qquad\qquad \lim_{x\to 0}\frac{\cos(x)-1}{x}=0$$

These (especially the first) are useful for finding various other limits.

Example 10.4 Find $\displaystyle\lim_{x\to 0}\frac{\tan(x)}{x}$.

Just inserting $x=0$ results in $\frac{\tan(0)}{0}=\frac{0}{0}$, so we try a different approach:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(x)}{x} = \lim_{x\to 0}\frac{\frac{\sin(x)}{\cos(x)}}{x} = \lim_{x\to 0}\frac{\frac{\sin(x)}{\cos(x)}}{\frac{x}{1}} = \lim_{x\to 0}\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{x} &= \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)}\\ &= 1\cdot\frac{1}{\cos(0)} = 1.\end{aligned}$$

Sometimes a limit will not have the exact form of one in Theorem 10.2 but can be made to match with a little algebra. For instance, let's work out $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{x}$. Here the $2x$ in the sin is not the same as the x on the bottom, so this does not exactly match the familiar $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x}=1$. To see how to fix this it's helpful to emphasize the structure of this limit by replacing the x with a box $\blacksquare$ that could represent any expression:

$$\lim_{\blacksquare\to 0}\frac{\sin(\blacksquare)}{\blacksquare}=1.$$

This means $\dfrac{\sin(\blacksquare)}{\blacksquare}$ approaches 1 as the gray box approaches 0. Look again at the limit we're trying to evaluate: $\displaystyle\lim_{x\to 0}\frac{\sin(\boxed{2x})}{x}$. The x on the bottom does not match the $2x$ in the box. But we can make it match by multiplying the fraction by $1=\frac{2}{2}$ and factoring out the 2 on the top:

$$\lim_{x\to 0}\frac{\sin(\boxed{2x})}{x} = \lim_{x\to 0}\frac{2}{2}\cdot\frac{\sin(\boxed{2x})}{x} = 2\lim_{x\to 0}\frac{\sin(\boxed{2x})}{\boxed{2x}} = 2\cdot 1 = 2.$$

Here $\displaystyle\lim_{x\to 0}\frac{\sin(\boxed{2x})}{\boxed{2x}}=1$ because $x\to 0$ makes $\boxed{2x}\to 0$, hence $\dfrac{\sin(\boxed{2x})}{\boxed{2x}}\to 1$. In conclusion $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{x}=2$.

Note that the following approach (which gives the correct answer) is wrong because $\sin(2x) \neq 2\sin(x)$.

$$\lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\sin(x)}{x} = 2\lim_{x\to 0} \frac{\sin(x)}{x} = 2\cdot 1 = 2.$$

Example 10.5 Find $\displaystyle\lim_{h\to 3} \frac{\sin(h-3)}{h^2+2h-15}$.

Just dividing the limit of the top by the limit of the bottom results in $\dfrac{\sin(3-3)}{3^2+2\cdot 3-15} = \dfrac{0}{0}$, so we have to try something else. Factoring gives a match:

$$\lim_{h\to 3} \frac{\sin(h-3)}{h^2+2h-15} = \lim_{h\to 3} \frac{\sin(h-3)}{(h+5)(h-3)} = \lim_{h\to 3} \frac{1}{h+5}\,\frac{\sin(h-3)}{h-3}.$$

By Theorem 10.2 $\displaystyle\lim_{h\to 3} \frac{\sin(h-3)}{h-3} = 1$, because $h-3$ approaches 0 as $h \to 3$. Continuing the above calculation,

$$\lim_{h\to 3} \frac{1}{h+5}\,\frac{\sin(h-3)}{h-3} = \lim_{h\to 3} \frac{1}{h+5} \cdot \lim_{h\to 3} \frac{\sin(h-3)}{h-3} = \frac{1}{8}\cdot 1 = \frac{1}{8}.$$

Therefore $\displaystyle\lim_{h\to 3} \frac{\sin(h-3)}{h^2+2h-15} = \frac{1}{8}$.

Example 10.6 Find $\displaystyle\lim_{x\to 0} \frac{\sin(\pi x)}{\sin(3x)}$.

Blindly trying a limit law yields $\displaystyle\lim_{x\to 0} \frac{\sin(\pi x)}{\sin(3x)} = \frac{\lim_{x\to 0}\sin(\pi x)}{\lim_{x\to 0}\sin(3x)} = \frac{\sin(0)}{\sin(0)} = \frac{0}{0}$, so we need to follow a different path. This one gives a match to Theorem 10.2:

$$\lim_{x\to 0} \frac{\sin(\pi x)}{\sin(3x)} = \lim_{x\to 0} \frac{\pi}{3}\,\frac{\sin(\pi x)}{\pi x}\,\frac{3x}{\sin(3x)} = \frac{\pi}{3}\lim_{x\to 0}\frac{\dfrac{\sin(\pi x)}{\pi x}}{\dfrac{\sin(3x)}{3x}} = \frac{\pi}{3}\,\frac{\lim_{x\to 0}\dfrac{\sin(\pi x)}{\pi x}}{\lim_{x\to 0}\dfrac{\sin(3x)}{3x}} = \frac{\pi}{3}\cdot\frac{1}{1}.$$

Therefore $\displaystyle\lim_{x\to 0} \frac{\sin(\pi x)}{\sin(3x)} = \frac{\pi}{3}$.

If you can work the exercises you are ready to move on.

Exercises for Chapter 10

Find the limits.

1. $\lim_{x \to \frac{\pi}{2}} \cot(x)$

2. $\lim_{x \to 5\pi} \cos(x)$

3. $\lim_{h \to \frac{\pi}{4}} \tan(h)$

4. $\lim_{x \to 7} \cos\left(\frac{\pi x}{6}\right)$

5. $\lim_{x \to 3} \tan\left(\frac{5\pi}{x}\right)$

6. $\lim_{x \to \frac{\pi}{4}} (\cos(x) + 2\cot(x))$

7. $\lim_{x \to \frac{\pi}{3}} (7 + 2\cos(x))^{\frac{2}{3}}$

8. $\lim_{x \to \pi} \frac{\cos^2(x) - \cos(x)}{\cos(x) - 1}$

9. $\lim_{x \to \pi/4} \sqrt{3\tan(x) + 1}$

10. $\lim_{x \to \frac{7\pi}{4}} \cos(x)\sin(x)$

11. $\lim_{x \to 0} \frac{\cos^2(x) - \cos(x)}{\cos(x) - 1}$

12. $\lim_{x \to \pi} \frac{x\cos(x) + x}{\cos(x) + 1}$

13. $\lim_{x \to \pi/2} \frac{\cos^2(x)}{\sin^2(x) - 1}$

14. $\lim_{x \to \pi/2} \frac{\pi \sin x}{2x}$

15. $\lim_{h \to 0} \frac{\pi \sin h}{2h}$

16. $\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\sin(2\theta)}$

17. $\lim_{x \to 2} \frac{\sin(2x - 4)}{5x - 10}$

18. $\lim_{x \to 0} \frac{\tan x}{3x}$

19. $\lim_{x \to 1} \frac{\sin(2x - 2)}{x - 1}$

20. $\lim_{x \to 0} \frac{2x}{\sin(3x)}$

21. $\lim_{x \to 0} \frac{\sin(x)}{\sin(2x)}$

22. $\lim_{x \to 0} \frac{\tan(\pi x)}{2x}$

23. $\lim_{x \to 1} \frac{\sin(x^2 - 1)}{x - 1}$

24. $\lim_{\theta \to 0} \frac{1}{\theta}\tan(3\theta)$

25. $\lim_{\theta \to 0} \frac{1}{\theta\cot(4\theta)}$

26. $\lim_{\theta \to 0} \frac{\sec(\theta)}{\theta\csc(2\theta)}$

27. $\lim_{\theta \to 0} \frac{\frac{1}{\theta}\sin(5\theta)}{\cos(\theta)}$

28. $\lim_{x \to 0} \frac{\sin(7x)}{5x}$

29. $\lim_{x \to 2\pi} \frac{\sin(1 - \cos x)}{1 - \cos x}$

30. $\lim_{x \to 0} \frac{\sin(\sqrt{9x})}{\sqrt{x}}$

31. In Theorem 10.2 we interpret x as a *radian measure*. Show that if x were *degrees* (not radians) then $\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{\pi}{180}$. This is yet another reason we prefer radians over degrees. By measuring angles with radians, $\lim_{x \to 0} \frac{\sin(x)}{x}$ has the simplest possible value, namely 1.

10.3 Exercise Solutions for Chapter 10

1. $\lim_{x\to\frac{\pi}{2}} \cot(x) = \cot(\pi/2) = 0$

3. $\lim_{h\to\frac{\pi}{4}} \tan(h) = \tan(\pi/4) = 1$

5. $\lim_{x\to 3} \tan\left(\frac{5\pi}{x}\right) = \tan\left(\frac{5\pi}{3}\right) = -\sqrt{3}$

7. $\lim_{x\to\pi/3} (7+2\cos(x))^{\frac{2}{3}} = \left(\lim_{x\to\pi/3}(7+2\cos(x))\right)^{\frac{2}{3}} = \sqrt[3]{7+2\cos(\pi/3)}^{\,2} = \sqrt[3]{8}^{\,2} = 4$

9. $\lim_{x\to\pi/4} \sqrt{3\tan(x)+1} = \sqrt{\lim_{x\to\pi/4}(3\tan(x)+1)} = \sqrt{3\tan(\pi/4)+1} = \sqrt{3+1} = 2$

11. $\lim_{x\to 0} \frac{\cos^2(x)-\cos(x)}{\cos(x)-1} = \lim_{x\to 0} \frac{\cos(x)(\cos(x)-1)}{\cos(x)-1} = \lim_{x\to 0}\cos(x) = 1$

13. $\lim_{x\to\pi/2} \frac{\cos^2(x)}{\sin^2(x)-1} = \lim_{x\to\pi/2} \frac{\cos^2(x)}{-\cos^2(x)} = -1$ (because $\sin^2(x)-1 = -\cos^2(x)$)

15. $\lim_{h\to 0} \frac{\pi\sin h}{2h} = \frac{\pi}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{\pi}{2}\cdot 1 = \frac{\pi}{2}$

17. $\lim_{x\to 2} \frac{\sin(2x-4)}{5x-10} = \lim_{x\to 2}\frac{\sin(2x-4)}{5(x-2)} = \lim_{x\to 2}\frac{2\sin(2x-4)}{2\cdot 5(x-2)} = \frac{2}{5}\lim_{x\to 2}\frac{\sin(2x-4)}{2x-4} = \frac{2}{5}$

19. $\lim_{x\to 1} \frac{\sin(2x-2)}{x-1} = \lim_{x\to 1}\frac{2\sin(2x-2)}{2(x-1)} = 2\lim_{x\to 1}\frac{\sin(2x-2)}{2x-2} = 2\cdot 1 = 2$

21. $\lim_{x\to 0} \frac{\sin(x)}{\sin(2x)} = \lim_{x\to 0}\frac{\sin(x)}{2x}\frac{2x}{\sin(2x)} = \frac{1}{2}\lim_{x\to 0}\frac{\sin(x)}{x}\frac{2x}{\sin(2x)} = \frac{1}{2}\lim_{x\to 0}\frac{\frac{\sin(x)}{x}}{\frac{\sin(2x)}{2x}} = \frac{1}{2}\cdot\frac{1}{1} = \frac{1}{2}$

23. $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x\to 1}\frac{\sin\big((x+1)(x-1)\big)}{x-1} = \lim_{x\to 1}\frac{(x+1)\sin\big((x+1)(x-1)\big)}{(x+1)(x-1)}$

$= \lim_{x\to 1}(x+1)\frac{\sin(x^2-1)}{x^2-1} = \lim_{x\to 1}(x+1)\lim_{x\to 1}\frac{\sin(x^2-1)}{x^2-1} = (1+1)\cdot 1 = 2$

25. $\lim_{\theta\to 0} \frac{1}{\theta\cot(4\theta)} = \lim_{\theta\to 0}\frac{\tan(4\theta)}{\theta} = \lim_{\theta\to 0}\frac{\sin(4\theta)}{\theta\cos(4\theta)} = \lim_{\theta\to 0}\frac{4}{\cos(4\theta)}\frac{\sin(4\theta)}{4\theta} = \frac{4}{1}\cdot 1 = 4$

27. $\lim_{\theta\to 0} \frac{\frac{1}{\theta}\sin(5\theta)}{\cos(\theta)} = \lim_{\theta\to 0}\frac{5}{\cos(\theta)}\frac{\sin(5\theta)}{5\theta} = \frac{5}{1}\cdot 1 = 5$

29. $\lim_{x\to 2\pi} \frac{\sin(1-\cos x)}{1-\cos x} = 1$ (because $1-\cos x \to 0$ as $x\to 2\pi$)