Section 5.5 Substitution Rule

Goal: Run the chain rule in reverse.

> Recall the Fundamental Theorem of Calculus.
>
> $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$

This means that computing a definite integral reduces to finding an indefinite integral, or antiderivative. We thus return to the problem of finding antiderivatives. So far we have the following formulas for doing this:

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad \int \sin u\,du = -\cos u + C \qquad \int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$$

$$\int \frac{1}{u}\,du = \ln|\ | + C \qquad \int \cos u\,du = \sin u + C \qquad \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$$

$$\int e^u\,du = e + C \qquad \int \sec^2 u\,du = \tan u + C \qquad \int \frac{1}{u\sqrt{u^2-1}}\,dx = \sec^{-1}|u| + C$$

$$\int a^u\,du = \frac{1}{\ln a} a + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = \csc u + C$$

This is an impressive list, but its by no means complete. Today we investigate a technique called integration by substitution that gives the list greater versitility. Basically it's the chain rule in reverse, so let's start there.

Suppose $F'(x) = f(x)$, i.e. $F(x)$ is an antiderivative of $f(x)$.

Chain Rule .... $\frac{d}{dx}\left[F(g(x))\right] = f(g(x))\,g'(x)$

Chain rule in reverse: $\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$

Let $u = g(x)$

$\frac{du}{dx} = g'(x)$

$du = g'(x)\,dx$

$$\int f(u)\,du = F(u) + C$$

In summary: Theorem 5.6

> Substitution Rule If $u = g(x)$, then $\underbrace{\int f(g(x))\,g'(x)\,dx}_{\text{complex integral}} = \underbrace{\int f(u)\,du}_{\text{simple integral.}}$

This rule can reduce a complex integral to a simple integral. To use it, you must look for the structure $f(g(x))\,g'(x)$ in the original integral. Then you simplify it by making the substitutions $g(x) = u$ and $g'(x)\,dx = du$. Some examples are in order.

$\underline{\text{Ex}}$ $\int \cos(\underbrace{x^2+3}_{u})\, \underbrace{2x\, dx}_{du}$

$= \int \cos(u)\, du$

$= \sin(u) + C = \boxed{\sin(x^2+3) + C}$

$u = x^2+3$
$\frac{du}{dx} = 2x$
$du = 2x\,dx$

Check:
$\frac{d}{dx}\left[\sin(x^2-3)\right] = \cos(x^2-3)\,2x$ ✓

$\underline{\text{Ex}}$ $\int e^{\sin x} \cos x \, dx$

$= \int e^u \, du$

$= e^u + C = \boxed{e^{\sin x} + C}$

$u = \sin x$
$\frac{du}{dx} = \cos x$
$du = \cos x \, dx$

Check:
$\frac{d}{dx}\left[e^{\sin x}\right] = e^{\sin x} \cos x$

$\underline{\text{Ex}}$ $\int \tan^5 x \sec^2 x \, dx$

$= \int u^5 \, du$

$= \frac{u^6}{6} + C = \boxed{\frac{\tan^6 x}{6} + C}$

$u = \tan x$
$\frac{du}{dx} = \sec^2 x$
$du = \sec^2 x \, dx$

$\underline{\text{Ex}}$ $\int \cos(x^2+3)\, x \, dx$

$u = x^2+3$
$du = 2x\,dx$

$= \frac{1}{2}\int \cos(x^2+3)\, 2x\, dx$

$= \frac{1}{2}\int \cos u \, du = \frac{1}{2}\sin u + C = \boxed{\frac{1}{2}\sin(x^2+3) + C}$

$\underline{\text{Ex}}$ $\int \sin(\pi x)\, dx = \frac{1}{\pi}\int \sin(\pi x)\, \pi \, dx = \frac{1}{\pi}\int \sin(u)\, du = -\frac{1}{\pi}\cos u + C$

$= \boxed{-\frac{1}{\pi}\cos(\pi x) + C}$

$u = \pi x$
$\frac{du}{dx} = \pi$
$du = \pi\, dx$

$\underline{\text{Ex}}$ $\int \frac{4x^3+1}{5x^4+5x-3}\, dx = \int \frac{1}{5x^4+5x+1}(4x^3+1)\,dx = \frac{1}{5}\int \frac{1}{5x^4+5x+1} 5(5x^3+1)\,dx$

$u = 5x^4+5x-3$

$\frac{du}{dx} = 20x^3+5$

$du = (20x^3+5)\,dx$
$= 5(5x^3+1)\,dx$

$= \frac{1}{5}\int \frac{1}{u}\, du$

$= \frac{1}{5}\ln|u| + C$

$= \boxed{\frac{1}{5}\ln|5x^4+5x-3| + C}$

Note Expression such as $\int \frac{1}{u}\,du$ often written $\int \frac{du}{u}$

Ex $$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{1}{\sqrt{1-(e^x)^2}}\,e^x\,dx$$

similar to $\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + c$

Thus: $u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x dx$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C$$

$$= \boxed{\sin^{-1}(e^x) + C}$$

Ex $$\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx = \int \sec^2(\sqrt{x})\,\frac{1}{\sqrt{x}}\,dx$$

$u = \sqrt{x} = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$= 2\int \sec^2(\sqrt{x})\,\frac{1}{2\sqrt{x}}\,dx = 2\int \sec^2 u\,du$$

$$= 2\tan u + C = \boxed{2\tan\sqrt{x} + C}$$

Ex $$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{\sin x}\cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\sin x| + C}$$

Its *essential* that you work *lots* of exercises for practice. You will get very good at this technique with practice. *Guaranteed*.

R. H.