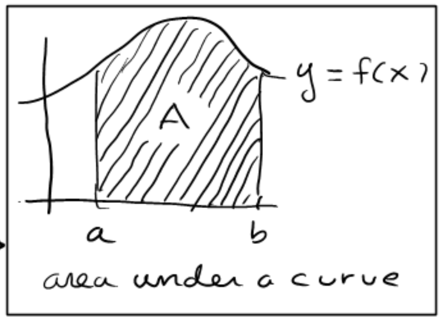
Chapter 5 Integration

So far, differentiation has been our main topic. Now we turn to our second major topic of the course: Integration.

In Section 4.9 we learned about antiderivatives. In Sections 5.1 and 5.2 we'll study a seemingly unrelated topic: area under a curve. Then, in Section 5.3, we'll learn about the Fundamental Theorem of Calculus which links antiderivatives and area.

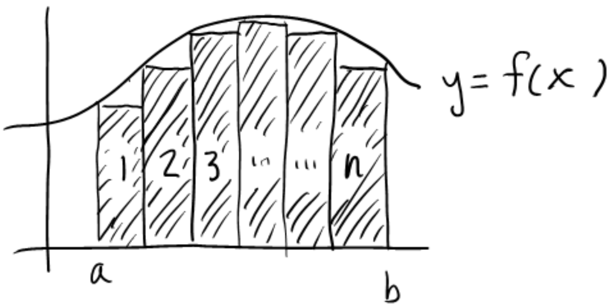
$\int f(x)\,dx$ antiderivatives	$\longleftrightarrow$	area under a curve
Section 4.9	Section 5.3 Fundamental Theorem of Calculus	Sections 5.1, 5.2



Plan: Today: Section 5.1: Area and sigma notation
Then 5.2 — Section 5.3 will tie all of this together.

Section 5.1 Area

In 5.1 and 5.2 we compute area under a curve by dividing the region into rectangles whose areas we can compute.



$$\text{Area under curve} \approx \begin{pmatrix}\text{sum of areas of}\\ n \text{ rectangles}\end{pmatrix}$$

$$\text{Then Area} = \lim_{n\to\infty}\begin{pmatrix}\text{sum of areas of}\\ n \text{ rectangle}\end{pmatrix}$$ ← we'll return to this soon.

To carry out this plan, we'll need a notation to express the sum of n numbers. That's what Section 5.2 is all about: Sigma Notation.

If f is a function, then:
$$\sum_{k=1}^{n} f(k) = f(1)+f(2)+f(3)+\dots+f(n)$$

Also, e.g.,
$$\sum_{k=3}^{6} f(k) = f(3)+f(4)+f(5)+f(6)$$
etc.

Ex $\displaystyle\sum_{k=1}^{5}(k^2+1) = (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)+(5^2+1) = 2+5+10+17+26 = \boxed{60}$

Ex $\displaystyle\sum_{k=0}^{10} k = 0+1+2+3+4+5+6+7+8+9+10 = \boxed{55}$

Ex $\displaystyle\sum_{k=1}^{1000} \sin(k\pi) = 0+0+0+\dots+0 = \boxed{0}$

Ex $\displaystyle\sum_{k=1}^{10} 2 = 2+2+2+2+2+2+2+2+2+2 = \boxed{20}$

Rules

① $\displaystyle\sum_{k=1}^{n} \left(f(k) + g(k)\right) = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$

② $\displaystyle\sum_{k=1}^{n} \left(f(k) - g(k)\right) = \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n} g(k)$

③ $\displaystyle\sum_{k=1}^{n} c f(k) = c \cdot \sum_{k=1}^{n} f(k)$ ← where c is a constant

Theorem 5.1

① $\displaystyle\sum_{k=1}^{n} c = nc$ ← Reason: $\displaystyle\sum_{k=1}^{n} c = \sum_{k=1}^{n} (0 \cdot k + c) = \underbrace{c + c + c + \cdots + c}_{k \text{ times}} = nc$

② $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ ← see text for a justification

③ $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

④ $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

You don't need to remember these

Ex $\displaystyle\sum_{k=1}^{10} k = 1 + 2 + 3 + \cdots + 10 = \frac{10(10+1)}{2} = \frac{10 \cdot 11}{2} = \frac{110}{2} = \boxed{55}$

agrees with what we got on previous page.

Ex $\displaystyle\sum_{k=1}^{100} \left(\frac{5 + 2k^2}{3}\right) = \sum_{k=1}^{100} \left(\frac{5}{3} + \frac{2}{3}k^2\right)$

$\displaystyle = \sum_{k=1}^{100} \frac{5}{3} + \sum_{k=1}^{100} \frac{2}{3}k^2$ (rule ①)

$\displaystyle = 100 \cdot \frac{5}{3} + \frac{2}{3} \sum_{k=1}^{100} k^2$ (rules ④, ③)

$\displaystyle = \frac{500}{3} + \frac{2}{3} \cdot \frac{100(100+1)(2 \cdot 100 + 1)}{6}$

$\displaystyle = \frac{500}{3} + \frac{200(101)(201)}{18}$

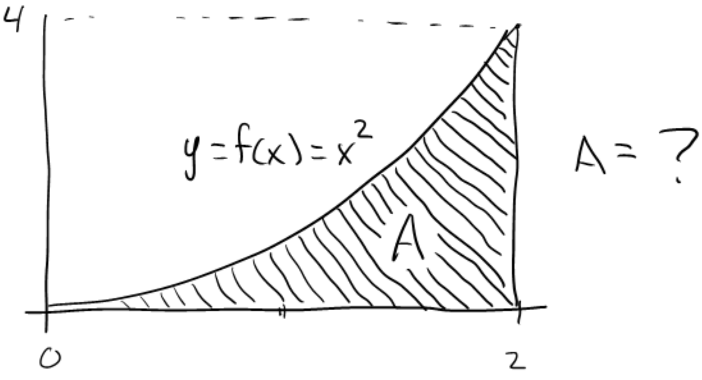
$\displaystyle \frac{3000}{18} + \frac{4060200}{18} = \boxed{\frac{677200}{3}}$

Now that we've developed sigma notation, lets use it to find the area under a curve.

__Recall__ $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\displaystyle\sum_{k=1}^{n} c f(k) = c \sum_{k=1}^{n} f(k)$

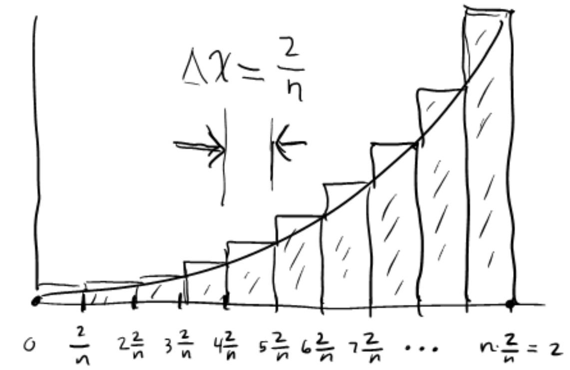
__Example__

Find the area under $y = f(x) = x^2$ between $x = 0$ and $x = 2$



Begin by dividing $[0,2]$ into n pieces, each of length $\Delta x = \frac{2}{n}$

On each interval, make a rectangle, as illustrated

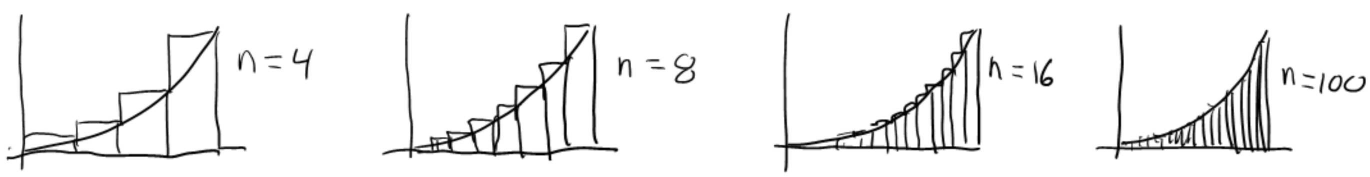


Area of rectangle #k is $\underbrace{f\left(k\cdot\frac{2}{n}\right)}_{\text{height}}\underbrace{\Delta x}_{\text{base}} = \left(\frac{2k}{n}\right)^2\frac{2}{n} = \frac{8k^2}{n^3}$

$$\begin{pmatrix}\text{Sum of areas of} \\ n \text{ rectangles}\end{pmatrix} = \sum_{k=1}^{n} f\left(k\cdot\frac{2}{n}\right)\Delta x = \sum_{k=1}^{n}\left(\frac{2k}{n}\right)^2\frac{2}{n} = \sum_{k=1}^{n}\frac{8k^2}{n^3} =$$

$$= \frac{8}{n^3}\sum_{k=1}^{n} k^2 = \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{4(n+1)(2n+1)}{3} = \boxed{\frac{4}{3}\,\frac{2n^2+3n+1}{n^2}}$$

For better approx make # of rectangles n larger.



$$A = \lim_{n\to\infty}\begin{pmatrix}\text{sum of areas} \\ \text{of } n \text{ rectangles}\end{pmatrix} = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\Delta x$$

$$= \lim_{n\to\infty}\frac{4}{3}\,\frac{2n^2+3n+1}{n^2} = \frac{4}{3}\cdot 2 = \boxed{\frac{8}{3} \text{ square units}}$$