Section 3.7 The Chain Rule

Let's begin with a summary of what we know so far.

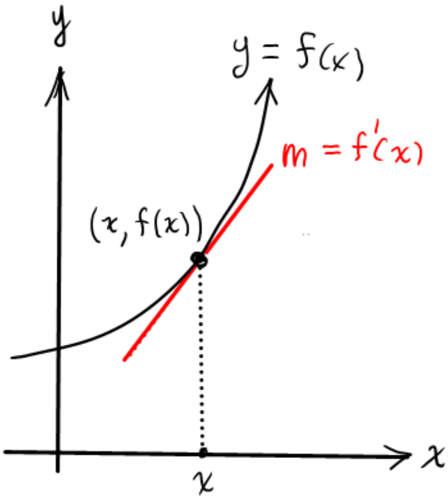
Definition

The <u>derivative</u> of a function $f(x)$ is another function $f'(x)$ defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$$= \left(\begin{array}{l} \text{slope of tangent to } y = f(x) \\ \text{at the point } (x, f(x)). \end{array} \right)$$



Here is our list of rules (so far) for finding derivatives without limits.

$\frac{d}{dx}[c] = 0$

$\frac{d}{dx}[x] = 1$

$\frac{d}{dx}[x^n] = nx^{n-1}$

$\frac{d}{dx}[e^x] = e^x$

$\frac{d}{dx}[\sin(x)] = \cos(x)$

$\frac{d}{dx}[\cos(x)] = -\sin(x)$

$\frac{d}{dx}[\tan(x)] = \sec^2(x)$

$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$

$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

$\frac{d}{dx}[cf(x)] = cf'(x)$

$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Today we are going to add a new rule to this list – the so-called <u>chain rule</u>. To motivate it, consider this problem:

$$\frac{d}{dx}\left[\sin(\underbrace{x^3 + x})\right] = \ ?$$

$$\frac{d}{dx}\left[f(g(x))\right] = \ ?$$

<u>Today's goal</u>: Rule for $\frac{d}{dx}[f(g(x))]$, i.e. $\frac{d}{dx}[f \circ g(x)]$. New rule is <u>chain rule</u>

To derive it we will use this version of the derivative: $f'(x) = \lim_{z \to x} \frac{f(z) - f(w)}{z - w}$

Helpful to look at it this way: $f'(\bigcirc) = \lim_{\square \to \bigcirc} \frac{f(\square) - f(\bigcirc)}{\square - \bigcirc}$

Let's go:

$$\frac{d}{dx}\left[f(g(x))\right] = \lim_{z\to x} \frac{f(g(z)) - f(g(x))}{z - x}$$

$$= \lim_{z\to x} \frac{f(g(z)) - f(g(x))}{g(z) - g(x)} \, \frac{g(z) - g(x)}{z - x}$$

$$= \lim_{z\to x} \frac{f(\boxed{g(z)}) - f(g(x))}{\boxed{g(z)} - g(x)} \, \frac{g(z) - g(x)}{z - x}$$

$$= f'(g(x)) \cdot g'(x)$$

$$= f'(g(x))\, g'(x)$$

$$\boxed{\text{Conclusion: } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)}$$

Another way of looking at it:

If $y = f(g(x))$, then $\begin{cases} y = f(u) \\ u = g(x) \end{cases}$

Then $\underbrace{\frac{d}{dx}\left[f(g(x))\right]}_{\frac{dy}{dx}} = f'(g(x))\, g'(x) = \underbrace{f'(u)\, g'(x)}_{\frac{dy}{du} \cdot \frac{du}{dx}}$

$$\frac{dy}{dx} = \cdots \quad \cdots = \frac{dy}{du} \cdot \frac{du}{dx}$$

Now let's sum all of this up.

The Chain Rule (for the derivative of a composition)

Version 1 If $y = f(g(x))$ then $\begin{cases} y = f(u) \\ u = g(x) \end{cases}$

and $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

Version 2 $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$

Now we can solve our motivational problem from the previous page

$$\frac{d}{dx}\left[\sin(x^3 + x)\right] = \cos(x^3 + x)\,(3x^2 + 1)$$

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$$

using version 2

Now let's solve this same problem with Version 1.

Suppose $y = \sin(x^3+x)$ $\begin{cases} y = \sin(u) \\ u = x^3+x \end{cases}$ Then: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \cos(u)(3x^2+1)$

$= \boxed{\cos(x^3+x)(3x^2+1)}$

Example Differentiate $y = \tan\left(\frac{1}{x}\right) = \tan(x^{-1})$

Version 1 $\begin{cases} y = \tan(u) \\ u = x^{-1} \end{cases}$ Thus $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \sec^2(u)(-x^{-1-1})$

$= \sec^2(x^{-1})(-x^{-2})$

$= -\sec^2\left(\frac{1}{x}\right)\frac{1}{x^2} = \boxed{\frac{-\sec^2\left(\frac{1}{x}\right)}{x^2}}$

Version 2 $\frac{d}{dx}\left[\tan(x^{-1})\right] = \sec^2(x^{-1})(-x^{-2}) = \boxed{-\frac{\sec^2\left(\frac{1}{x}\right)}{x^2}}$

Example Differentiate $y = \sec\sqrt{x}$

Version 1 $\begin{cases} y = \sec u \\ u = \sqrt{x} = x^{1/2} \end{cases}$ Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \sec(u)\tan(u)\cdot\frac{1}{2}x^{\frac{1}{2}-1}$

$= \sec(\sqrt{x})\tan(\sqrt{x})\,\frac{1}{2}x^{-\frac{1}{2}}$

$= \boxed{\frac{\sec(\sqrt{x})\tan(\sqrt{x})}{2\sqrt{x}}}$

Version 2 $\frac{d}{dx}\left[\sec(x^{\frac{1}{2}})\right] = \sec(x^{\frac{1}{2}})\tan(x^{\frac{1}{2}})\frac{1}{2}x^{-\frac{1}{2}} = \boxed{\frac{\sec(\sqrt{x})\tan(\sqrt{x})}{2\sqrt{x}}}$

Compositions involving more than two functions:
The chain rule extends as follows:

If $y = f(g(h(x)))$, then $\begin{cases} y = f(u) \\ u = g(z) \\ z = g(x) \end{cases}$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx}$

Example Differentiate $y = \tan(\cos(x^2))$

Version 1 $\begin{cases} y = \tan(u) \\ u = \cos(z) \\ z = x^2 \end{cases}$ $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx} = \sec^2(u)(-\sin(z))\,2x$

$= \sec^2(\cos(z))(-\sin(x^2))\,2x$

$= \sec^2(\cos(x^2))(-\sin(x^2))\,2x$

$= \boxed{-2x\sec^2(\cos(x^2))\sin(x^2)}$

Version 2 $\frac{d}{dx}\Big[\overset{f}{\tan}\big(\overbrace{\cos(x^2)}^{g(x)}\big)\Big] = \sec^2(\cos(x^2))\,\frac{d}{dx}\left[\cos(x^2)\right]$

$= \sec^2(\cos(x^2))(-\sin(x^2)\,2x)$

Example $y = \sin^5(x) = (\sin(x))^5$.

Find $\frac{dy}{dx}$. Solution $\begin{cases} y = u^5 \\ u = \sin(x) \end{cases}$ Thus: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$= 5u^4\cos(x) = \boxed{5(\sin(x))^4\cos(x)}$$

After working with the chain rule for awhile, it becomes second nature. One pattern that evolves is that the chain rule extends our basic rules:

$$\frac{d}{dx}\left[\sin(g(x))\right] = \cos(g(x))\,g'(x)$$

$$\frac{d}{dx}\left[\cos(g(x))\right] = -\sin(g(x))\,g'(x)$$

$$\frac{d}{dx}\left[\tan(g(x))\right] = \sec^2(g(x))\,g'(x)$$

$$\frac{d}{dx}\left[\cot(g(x))\right] = -\csc^2(g(x))\,g'(x)$$

$$\frac{d}{dx}\left[\sec(g(x))\right] = \sec(g(x))\tan(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left[\csc(g(x))\right] = -\csc(g(x))\cot(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left[(g(x))^n\right] = n(g(x))^{n-1}\,g'(x)$$ ← "generalized power rule"

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}\,g'(x)$$

Examples

ⓐ $\frac{d}{dx}\left[5e^{\tan(x)}\right] = 5\frac{d}{dx}\left[e^{\tan(x)}\right] = \boxed{5e^{\tan(x)}\sec^2(x)}$

ⓑ $\frac{d}{dx}\left[(\tan(x))^{10}\right] = 10(\tan(x))^9\sec^2(x) = \boxed{10\tan^9(x)\sec^2(x)}$

ⓒ $\frac{d}{dx}\left[(\tan(2x^3))^{10}\right] = 10(\tan(2x^3))^9\frac{d}{dx}\left[\tan(2x^3)\right]$

$$= \boxed{10\tan^9(2x^3)\sec^2(2x^3)\,6x^2}$$

ⓓ $\frac{d}{dx}\left[(x^3+x)\cos\left(\frac{x^3}{x^2-1}\right)\right] = (3x^2+1)\cos\left(\frac{x^3}{x^2-1}\right) + (x^3+x)\frac{d}{dx}\left[\cos\left(\frac{x^3}{x^2-1}\right)\right]$

$$= \boxed{(3x^2+1)\cos\left(\frac{x^3}{x^2-1}\right) + (x^3+x)\left(-\sin\left(\frac{x^3}{x^2-1}\right)\right)\frac{3x^2(x^2-1)-x^3(2x-0)}{(x^2-1)^2}}$$

Example

$$\frac{d}{dx}\left[\sqrt{x^3\sin(x) + \cos(x)}\right]$$

$$= \frac{d}{dx}\left[\left(x^3\sin(x) + \cos(x)\right)^{\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left(x^3\sin(x) + \cos(x)\right)^{\frac{1}{2}-1} \frac{d}{dx}\left[x^3\sin(x) + \cos(x)\right]$$

$$= \frac{1}{2}\left(x^3\sin(x) + \cos(x)\right)^{-\frac{1}{2}} \left(3x^2\sin(x) + x^3\cos(x) - \sin(x)\right)$$

$$= \boxed{\frac{3x^2\sin(x) + x^3\cos(x) - \sin(x)}{2\sqrt{x^3\sin(x) + \cos(x)}}}$$

VERY IMPORTANT

Get lots of practice using the chain rule (and other rules).

We will build on these techniques, so we need to be fluent with them.

Skill improves quickly with practice.

But practice involves working lots of exercises

— Richard