# Section 3.4 The Product and Quotient Rules

## Recall

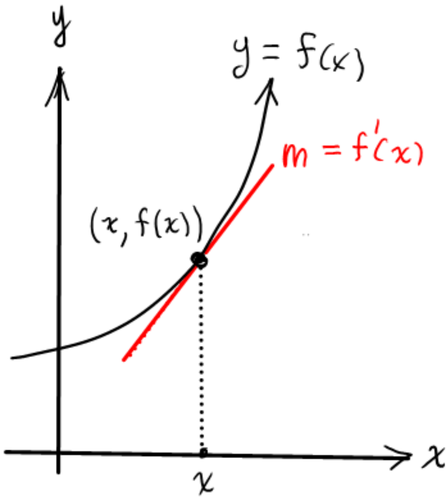
**Definition**

The derivative of a function $f(x)$ is another function $f'(x)$ defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

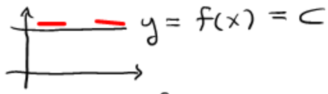
$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$$= \left( \begin{array}{l} \text{slope of tangent to } y = f(x) \\ \text{at the point } (x, f(x)). \end{array} \right)$$

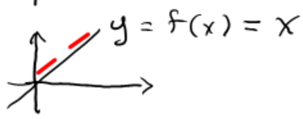



Last time we deduced rules for computing some derivatives

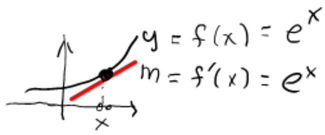
- $\frac{d}{dx}[c] = 0$ ............ (constant rule)




- $\frac{d}{dx}[x] = 1$ ............ (identity rule)




- $\frac{d}{dx}[x^n] = n x^{n-1}$ ............ (power rule)
- $\frac{d}{dx}[e^x] = e^x$ ............




- $\frac{d}{dx}\left[c f(x)\right] = c f'(x)$ ............ (constant multiple rule)
- $\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$ ... (sum/difference rule)

**Important points**

- Derivative was defined with a limit because the limit gives slope of tangent line. Limit definition of $f'(x)$ gives meaning to $f'(x)$. Rules are short-cuts, but give no meaning.

- If $f(x)$ does not match rules we may have to go back to limit to find $f'(x)$.

**Today's goal:**

New rules: $\begin{cases} \frac{d}{dx}\left[f(x)g(x)\right] = \,? \\ \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \,? \end{cases}$

$$\frac{d}{dx}\left[ f(x)g(x) \right] = ?$$

Tempting: $\cancel{\frac{d}{dx}\left[ f(x)g(x) \right] = f'(x)g'(x)}$ ← But WRONG

Because $f(x)g(x)$ fits no rule, we have to use the limit definition.

$$\frac{d}{dx}\left[ f(x)g(x) \right] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\left(g(x+h) - g(x)\right) + \left(f(x+h) - f(x)\right)g'(x)}{h}$$

$$= \lim_{h\to 0} \left( f(x+h)\frac{g(x+h)-g(x)}{h} + \frac{f(x+h)-f(x)}{h}g(x) \right)$$

$$= f(x+0)\, g'(x) + f'(x)g(x)$$

$$= f(x)g'(x) + f'(x)g(x)$$

Conclusion

$$\boxed{\text{Product Rule} \quad \frac{d}{dx}\left[ f(x)g(x) \right] = f'(x)g(x) + f(x)g'(x)}$$

Examples:

- $\frac{d}{dx}\left[ (x^3+3)(4x^2+x) \right] = (3x^2+0)(4x^2+x) + (x^3+3)(8x+1)$

  $= 12x^4 + 3x^3 + 8x^4 + x^3 + 24x + 3 = \boxed{20x^4 + 4x^3 + 24x + 3}$

- $\frac{d}{dx}\left[ (x^3+3)(4x^2+x) \right] = \frac{d}{dx}\left[ 4x^5 + x^4 + 12x^2 + 3x \right] = \boxed{20x^4 + 4x^3 + 24x + 3}$

(Alternate approach – there will often be several ways to find $f'(x)$)

- $\frac{d}{dx}\left[ 5x^{10} \right] = \frac{d}{dx}[5]x^{10} + 5\frac{d}{dx}\left[ x^{10} \right] = 0\cdot x^{10} + 5\cdot 10x^9 = \boxed{50x^9}$

- $\frac{d}{dx}\left[ 5x^{10} \right] = 5\frac{d}{dx}\left[ x^{10} \right] = 5\cdot 10x^9 = \boxed{50x^9}$ ← work enough of these problems so that you see the shorter approaches

- $\frac{d}{dx}\left[ e^x\sqrt{x} \right] = \frac{d}{dx}\left[ e^x x^{\frac{1}{2}} \right] = \frac{d}{dx}\left[ e^x \right] x^{\frac{1}{2}} + e^x\frac{d}{dx}\left[ x^{\frac{1}{2}} \right]$

  $= e^x\sqrt{x} + e^x \frac{1}{2}x^{-1/2} = \boxed{e^x\sqrt{x} + \frac{e^x}{2\sqrt{x}}}$

$$\boxed{\text{Quotient Rule} \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}$$

read the proof of this in the text!

Examples

- $$\frac{d}{dx}\left[\frac{x^2+1}{x^2-x}\right] = \frac{(2x+0)(x^2-x) - (x^2+1)(2x-0)}{(x^2-x)^2} = \frac{2x^3 - 2x^2 - 2x^3 - 2x}{(x^2-x)(x^2-x)} = \frac{2x^2-2x}{(x^2-x)(x^2-x)}$$

$$= \frac{x(2x-2)}{x(x-1)(x^2-x)} = \boxed{\frac{2x-2}{(x-1)(x^2-x)}}$$

- $$\frac{d}{dx}\left[\frac{\sqrt{x}}{e^x}\right] = \frac{d}{dx}\left[\frac{x^{\frac{1}{2}}}{e^x}\right] = \frac{\frac{1}{2}x^{-\frac{1}{2}}e^x - x^{\frac{1}{2}}e^x}{(e^x)^2} = \frac{\cancel{e^x}\left(\frac{x^{-1/2}}{2} - x^{\frac{1}{2}}\right)}{\cancel{e^x}e^x}$$

$$= \frac{\frac{1}{2x^{1/2}} - x^{\frac{1}{2}}}{e^x} = \boxed{\frac{\frac{1}{2\sqrt{x}} - \sqrt{x}}{e^x}}$$

- $$\frac{d}{dx}\left[\frac{x^2+x}{5}\right] = \frac{(2x+1)\cdot 5 - (x^2+x)\cdot 0}{5^2} = \frac{(2x+1)\cdot 5}{25} = \boxed{\frac{2x+1}{5}}$$

- $$\frac{d}{dx}\left[\frac{x^2+x}{5}\right] = \frac{d}{dx}\left[\frac{1}{5}(x^2+x)\right] = \frac{1}{5}\frac{d}{dx}\left[(x^2+x)\right] = \boxed{\frac{2x+1}{5}}$$

Work enough of these that you see the quicker approach

- $$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{0\cdot x - 1\cdot 1}{x^2} = \boxed{\frac{-1}{x^2}}$$

- $$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = -1\cdot x^{-1-1} = \boxed{-\frac{1}{x^2}}$$

can use quotient rule or power rule

- $$f(x) = 3x^5 + e^x x^2 - x + 4$$

$$f'(x) = 15x^4 + e^x x^2 + e^x 2x - 1 + 0$$

Note:
In one step we used all of our new rules, except the quotient rule.

Get used to using and combining these rules.
Practice – practice – practice !!

This section also introduces a rule $\frac{d}{dx}[e^{kx}] = ke^{kx}$.

You can ignore this rule because in a later section we'll learn a very powerful rule called the chain rule that will take care of this kind of problem.

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## Higher Order Derivatives

You can take derivatives of derivatives.:

$f(x) = x^4 + x^3 + x^2 + x + 1$

$f'(x) = 4x^3 + 3x^2 + 2x + 1$ ← (first derivative)

$f''(x) = 12x^2 + 6x + 2$ ← (2nd derivative)

$f'''(x) = 24x + 6$ ← (3rd derivative)

$f^{(4)}(x) = 24$ ← (4th derivative)

$f^{(5)}(x) = 0$ etc.

$f^{(6)}(x) = 0$

$\vdots$

$g(x) = \frac{1}{x}$

$g'(x) = -\frac{1}{x^2}$

$g''(x) = \frac{2}{x^3}$

$g'''(x) = -\frac{6}{x^4}$

$g^{(4)}(x) = \frac{24}{x^5}$

$\vdots$

Notation: If $y = f(x)$ then

$$f'(x) = y' = \frac{d}{dx}[f(x)] = \frac{d}{dx}[y] = \frac{dy}{dx}$$

$$f''(x) = y'' = \frac{d}{dx}\frac{d}{dx}[f(x)] = \frac{d^2}{dx^2}[f(x)] = \frac{d^2y}{dx^2}$$

$$f'''(x) = y''' = \frac{d}{dx}\frac{d^2}{dx^2}[f(x)] = \frac{d^3}{dx^3}[f(x)] = \frac{d^3y}{dx^3}$$

etc.