Section 3.2 Working with Derivatives

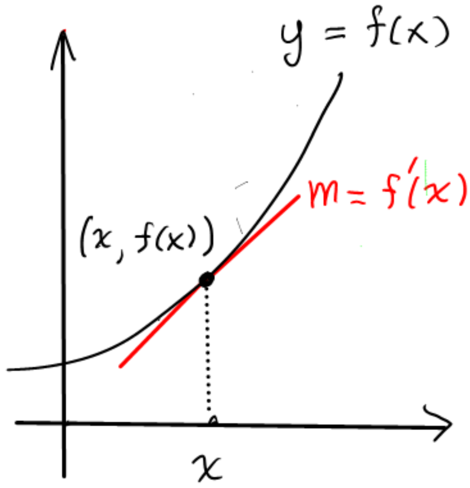
Last time we introduced a very fundamental idea: The derivative of a function

Definition

The derivative of a function $f(x)$ is another function called $f'(x)$ and defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \left(\begin{array}{l}\text{slope of tangent to} \\ y = f(x) \text{ at } (x, f(x))\end{array}\right)$$



Example $f(x) = \sqrt[3]{x}$. Find $f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

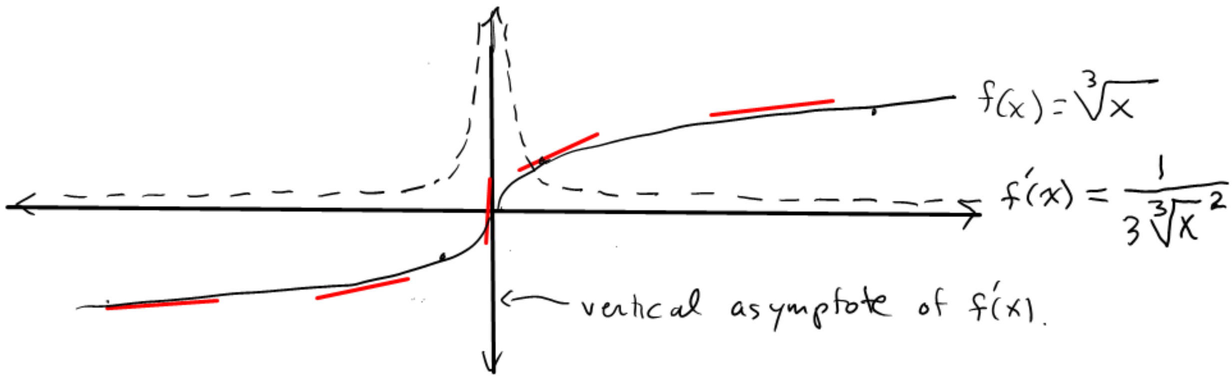
$$= \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} = \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \cdot \frac{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \sqrt[3]{x}^2}{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \sqrt[3]{x}^2}$$

$$= \lim_{h\to 0} \frac{x + h - x}{h\left(\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \sqrt[3]{x}^2\right)} = \lim_{h\to 0} \frac{1}{\sqrt[3]{x+h}^2 + \sqrt[3]{x+h}\sqrt[3]{x} + \sqrt[3]{x}^2}$$

$$= \frac{1}{\sqrt[3]{x+0}^2 + \sqrt[3]{x+0}\sqrt[3]{x} + \sqrt[3]{x}^2} = \frac{1}{3\sqrt[3]{x}^2}$$

Thus $\boxed{f'(x) = \frac{1}{3\sqrt[3]{x}^2}}$ $= \left(\begin{array}{l}\text{slope of tangent to } y = \sqrt[3]{x} \\ \text{at point } (x, f(x)) = (x, \sqrt[3]{x})\end{array}\right)$

Now let's compare the graphs of $f(x) = \sqrt[3]{x}$ and $f'(x) = \frac{1}{3\sqrt[3]{x}}$



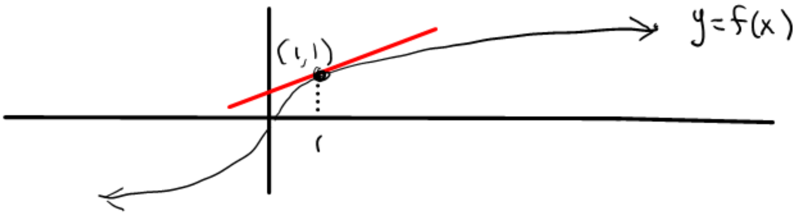
Important: $f'(x)$ equals slope of tangent to $f(x)$ at $(x, f(x))$

Where slope is steep (near $x=0$) $f'(x)$ is big. Where slope near 0, $f'(x)$ near 0.

Tangent to $y = f(x)$ is vertical at $(0,0)$ so $f'(0)$ is undefined.

Domain of $f(x)$ is $(-\infty, \infty)$ but domain of $f'(x)$ is $(-\infty, 0) \cup (0, \infty)$.

Example Find equation of tangent line to $f(x)=\sqrt[3]{x}$ at point $(1,\sqrt[3]{1})=(1,1)$

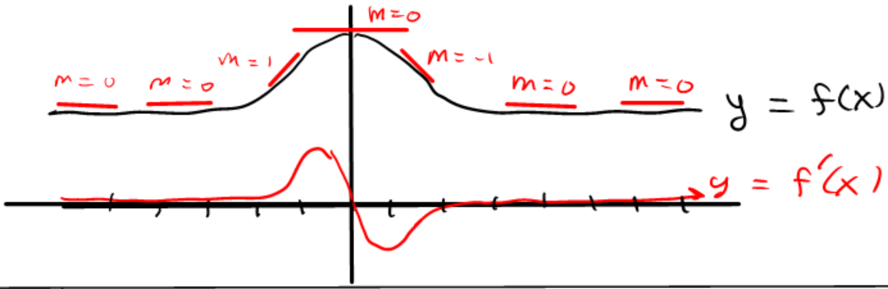


Slope of tangent: $f'(1)=\dfrac{1}{3\sqrt[3]{1}}=\dfrac{1}{3}$

Point-slope form:
$$y-y_0=m(x-x_0)$$
$$y-1=\tfrac{1}{3}(x-1)$$
$$y=\tfrac{1}{3}x-\tfrac{1}{3}+1$$

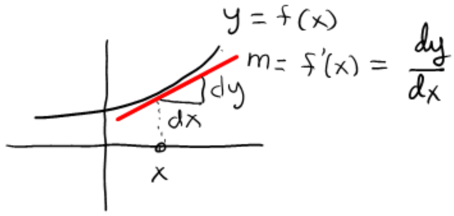
$$\boxed{\text{Answer} \quad y=\tfrac{1}{3}x+\tfrac{2}{3}}$$

Example The graph of $f(x)$ is given. Sketch the graph of $f'(x)$.



x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$f'(x)=$ (slope of $f(x)$ at $(x,f(x))$)	0	0	0	0,5	1	0	-1	0,5	0	0	0	0	0

Notations for the Derivative



If $y=f(x)$, then

$$f'(x)=y'=\frac{dy}{dx}=\frac{d}{dx}[f(x)]=\frac{d}{dx}(y)=D_x[y]=D_x[f(x)]$$

$$f'(a)=\left.\frac{dy}{dx}\right|_{x=a}$$

Example $y=\sqrt[3]{x}$ $\quad \dfrac{dy}{dx}=\dfrac{1}{3\sqrt[3]{x}^2}$ $\quad \left.\dfrac{dy}{dx}\right|_{x=1}=\dfrac{1}{3\sqrt[3]{1}}=\dfrac{1}{3}$

If $z=g(w)$, then

$$g'(w)=z'=\frac{dz}{dw}=\frac{d}{dw}[g(w)]=\frac{d}{dw}(z)=D_w[z]=D_w[g(w)]$$

$$g'(a)=\left.\frac{dz}{dw}\right|_{w=a}$$

Coming Next Rules for computing derivatives without limits!