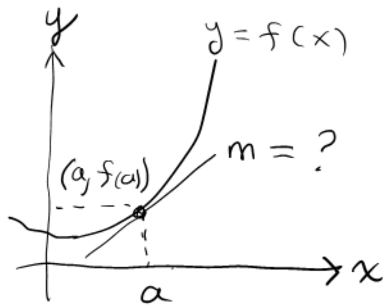
Part 3 Derivatives

A key problem of Calculus is to find the slope of the tangent to the graph of $y = f(x)$ at a point $(x, f(x))$

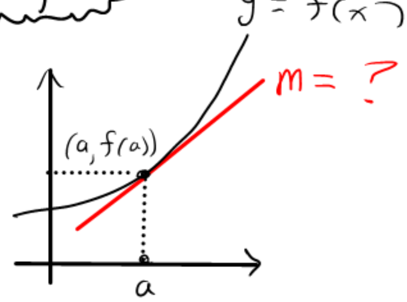


Erlier we saw that solving this problem seems to involve limits. This is the reason we studied limits in Part 2, and the reason for the text's sequencing.

Part 1 Function Review → Part 2 Limits of Functions → Part 3 Derivatives → Part 4 Applications of Derivatives

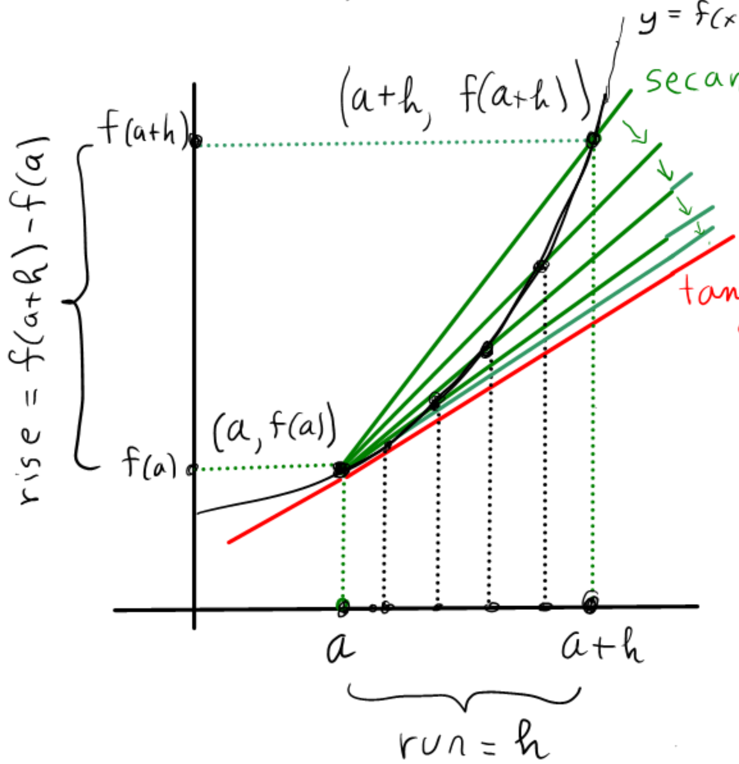
systematic method of finding slopes of tangents

Section 3.1 Introducing the Derivative



Fundamental Problem Find slope of tangent line to graph of $y=f(x)$ at point $(a, f(a))$

Strategy: Let h be a small number and draw a secant line through $(a, f(a))$ and $(a+h, f(a+h))$, as illustrated below.



secant $m = \frac{\text{rise}}{\text{run}} = \dfrac{f(a+h) - f(a)}{h}$

tangent slope $= \displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

As $h \to 0$, secant rotates toward tangent

As $h \to 0$ secant slope approaches tangent slope

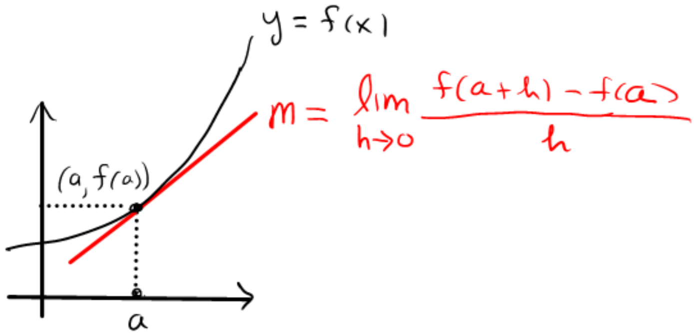
As $h \to 0$ $\dfrac{f(a+h) - f(a)}{h}$ approaches tangent slope

i.e. Tangent slope $= \displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ ☆

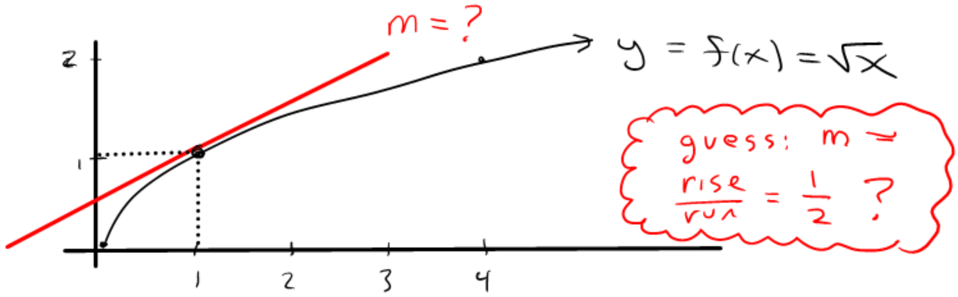
Conclusion

The slope of the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ is

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$



Example Find slope of tangent to $y = f(x) = \sqrt{x}$ at $(1, f(1)) = (1, \sqrt{1}) = (1, 1)$



$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

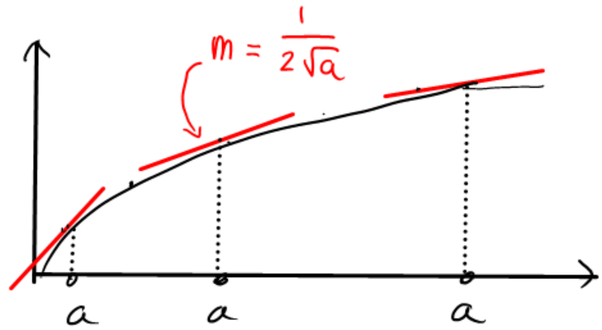
$$= \lim_{h\to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} = \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \, \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+h}^2 - 1^2}{h(\sqrt{1+h} + 1)} = \lim_{h\to 0} \frac{1 + h - 1}{h(\sqrt{1+h} - 1)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h} + 1)} = \lim_{h\to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{\sqrt{1+0}+1} = \boxed{\frac{1}{2}}$$

Let's redo this example, but instead of finding the slope at $(1, \sqrt{1}) = (1, 1)$ let's find the slope at $(a, \sqrt{a})$ where a is a (positive) unspecified number.

Slope m should then depend on a



$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

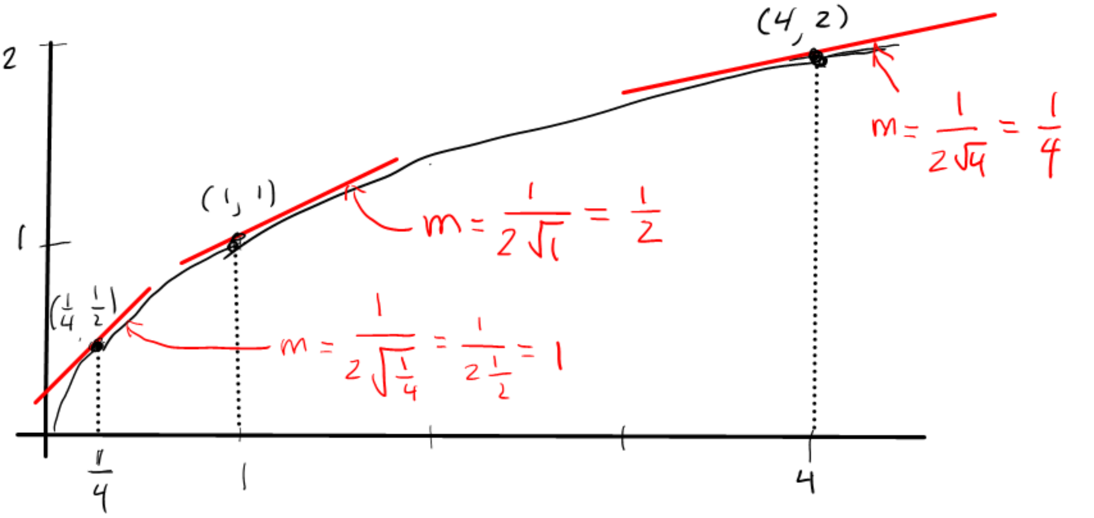
$$= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} = \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \, \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}$$

$$= \lim_{h\to 0} \frac{\sqrt{a+h}^2 - \sqrt{a}^2}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h\to 0} \frac{a + h - a}{h(\sqrt{a+h} - \sqrt{a})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{a+h} + \sqrt{a})} = \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{\sqrt{a+0} + \sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$$

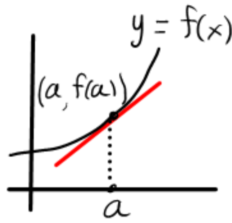
So the slope at $(a, f(a))$ is $\frac{1}{2\sqrt{a}}$

The slope $\frac{1}{2\sqrt{a}}$ depends on a, so it is a function of a.



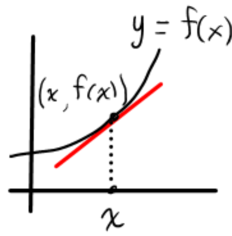
We may as well call a x and say slope at $(x, f(x))$ is $\frac{1}{2\sqrt{x}}$.

Slope of tangent to $y = f(x)$ at $(a, f(a))$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$



change a to x

Slope of tangent to $y = f(x)$ at $(x, f(x))$ is $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

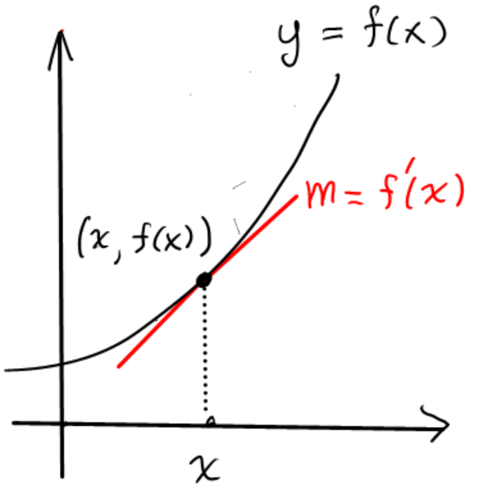


Definition

The derivative of a function $f(x)$ is another function called $f'(x)$ and defined as

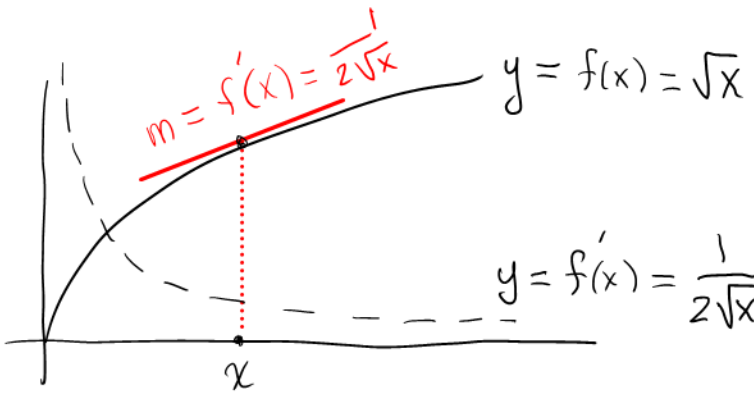
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \left(\begin{array}{l} \text{slope of tangent to} \\ y = f(x) \text{ at } (x, f(x)) \end{array} \right)$$



Example

The derivative of $f(x) = \sqrt{x}$ is the function $f'(x) = \frac{1}{2\sqrt{x}}$



Example

Find the derivative of $f(x) = x^2 + 2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{((x+h)^2+2)-(x^2+2)}{h}$$

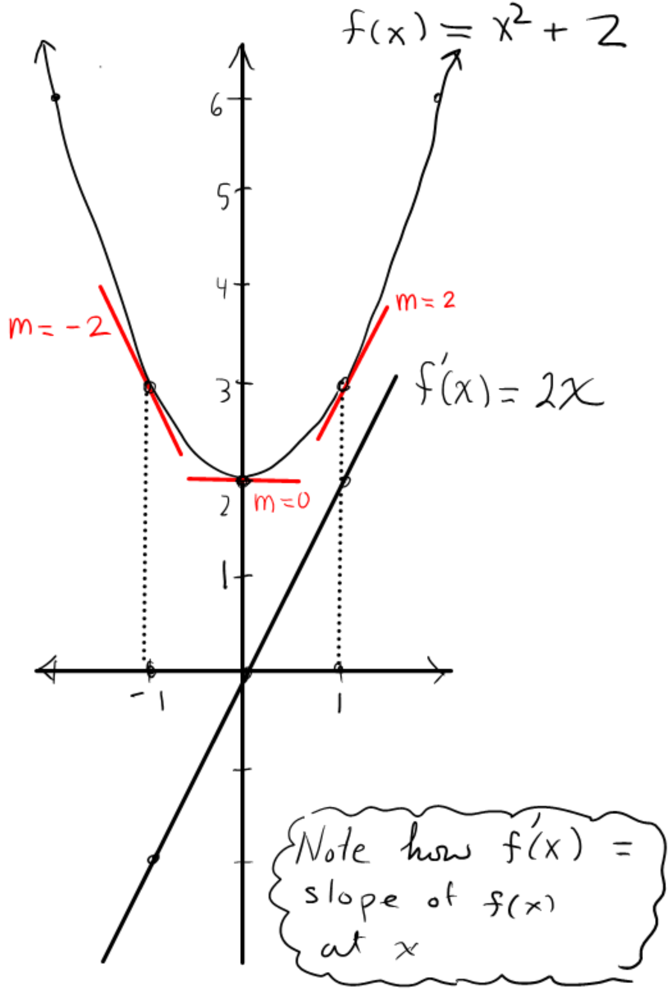
$$= \lim_{h\to 0} \frac{x^2+2xh+h^2+2-x^2-2}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (2x+h) = 2x+0$$

$= 2x.$ Therefore $\boxed{f'(x) = 2x}$



Example

Find the derivative of $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

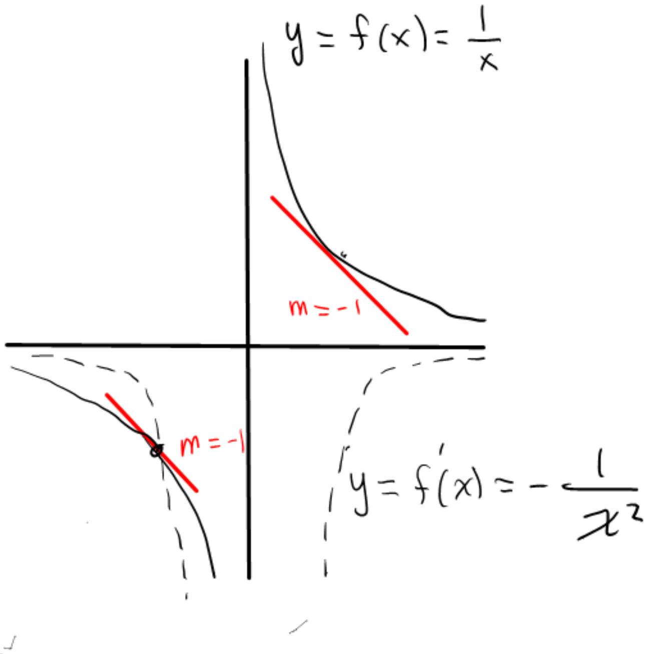
$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x}{x}\frac{1}{x+h} - \frac{1}{x}\frac{x+h}{x+h}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}} = \lim_{h\to 0} \frac{-h}{x(x+h)} \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$



Therefore the derivative of $f(x) = \frac{1}{x}$ is the function $\boxed{f'(x) = -\frac{1}{x^2}}$