Section 3.11 Related Rates

Sometimes in applications you will know one or more rates of change, and you'll need to find another rate of change. Today we discuss a method that accomplishes this.

First, let's review the rate-of-change interpretation of the derivative.

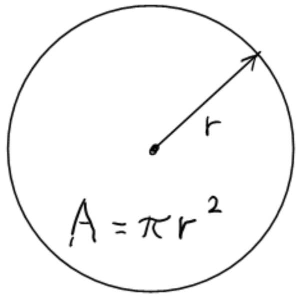
> If $f(t)$ is a function of time t,
> then $f'(t)$ is the rate of change of f at time t.
>
> Examples
>
> $y = P(t)$ = population of a city at time t (people)
>
> $\frac{dy}{dt} = P'(t)$ = rate of change in population at time t (people/year)
>
> $w(t)$ = amount of water in a tank at time t (gallons)
>
> $w'(t)$ = rate at which water is being added or removed. (gallons/min)

Now lets do a simple example that illustrates the kind of thinking involved in this section.

Example

Radius of a circle is increasing at a rate of 5 inches per second.

How quickly is the area changing when $r = 10''$?



$r = f(t)$
$A = g(t)$

Solution: Call the area A. So A and r are functions of time t.

Know: $\frac{dr}{dt} = 5$. Want $\frac{dA}{dt}$ (when $r = 10''$)

$$A = \pi r^2$$

$$\frac{d}{dt}[A] = \frac{d}{dt}\left[\pi r^2\right]$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt} = \pi 2r \cdot 5$$

$$= 10\pi r \text{ square inches per second.}$$

So when $r = 10$,

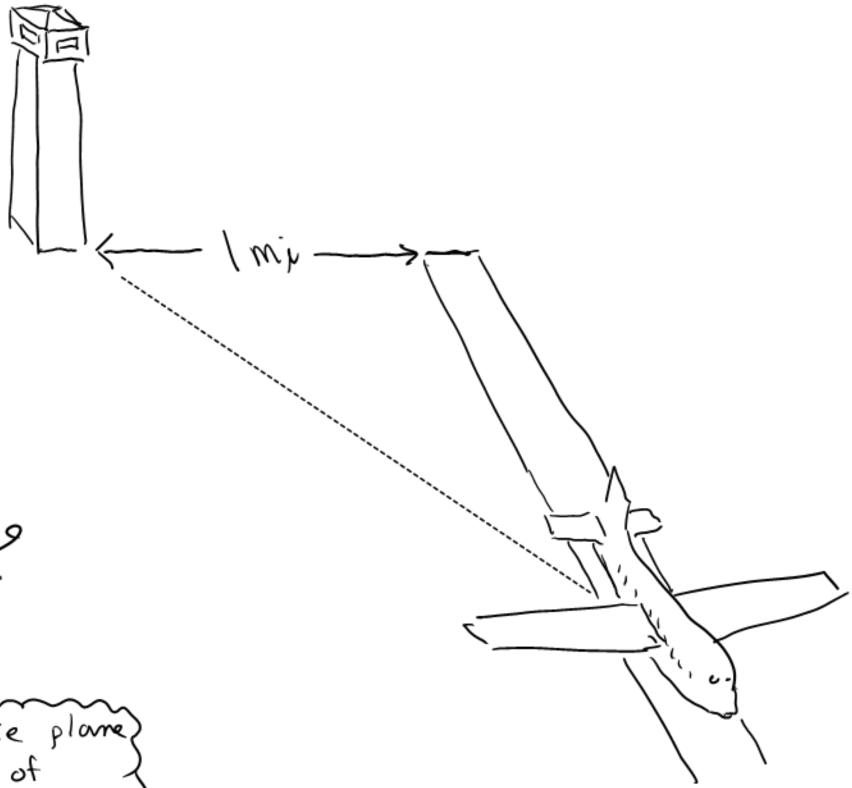
$$\frac{dA}{dt} = 10\pi 10 =$$

$$= \boxed{100\pi \text{ square inches per sec.}}$$

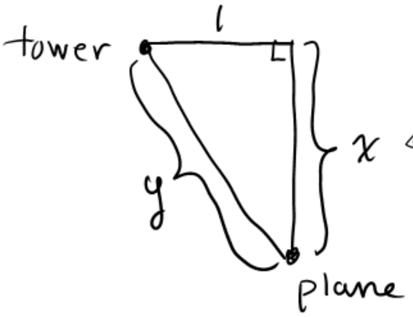
Example

Distance between plane and tower increases at a rate of 100 miles per hour.

How fast is plane moving when it's $5/3$ miles from tower?



To answer this, first summarize the situation with a diagram:



x is distance plane is from end of runway, so $\frac{dx}{dt}$ is plane's velocity

Then identify the rate you know and the rate you want.

Know $\frac{dy}{dt} = 100$ mph

Want $\frac{dx}{dt}$ (mph)

Next find an equation relating the quantities (x & y) whose rates are involved

$$1^2 + x^2 = y^2$$

$$\frac{d}{dt}\left[1 + x^2\right] = \frac{d}{dt}\left[y^2\right]$$

$$0 + 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

$$x\frac{dx}{dt} = y\frac{dy}{dt}$$

$$x\frac{dx}{dt} = y \cdot 100$$

$$\frac{dx}{dt} = \frac{100\,y}{x} \text{ mph}$$

Need $\frac{dx}{dt}$ when $y = 5/3$

$$x = \sqrt{\left(\frac{5}{3}\right)^2 - 1} = \sqrt{\frac{25}{9} - 1} = \sqrt{\frac{16}{9}} = \frac{4}{3}$$

Therefore $\frac{dx}{dt} = \frac{100\,y}{x} = \frac{100 \cdot \frac{5}{3}}{\frac{4}{3}}$

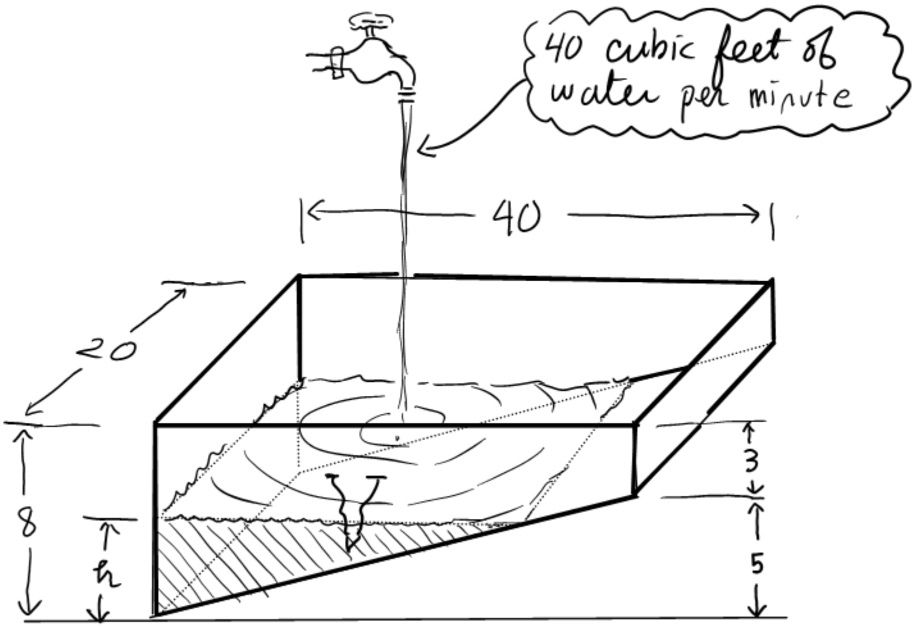
$= 125$ mph.

Answer Plane is going 125 mph when it's $5/3$ mile from tower.

Example

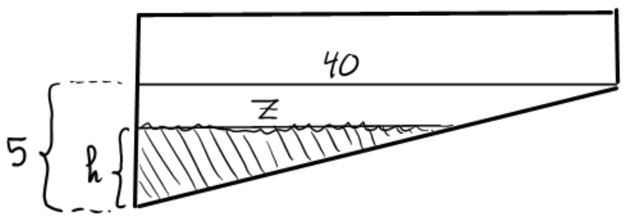
Pool is filling up. →

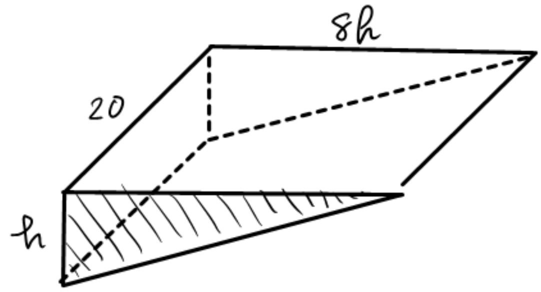
How fast is the water level h rising when it 3 feet deep (at deep end)?



Know $\dfrac{dV}{dt} = 40$

Want $\dfrac{dh}{dt}$ (when $h = 3$)





By similiar triangles,

$$\frac{z}{h} = \frac{40}{5}$$

and therefore $z = 8h$

Volume of water is

$$V = 20\left(\begin{array}{c}\text{area of}\\ \text{shaded } \Delta\end{array}\right)$$

$$V = 20\,\frac{1}{2}\,8h \cdot h$$

$$V = 80h^2$$

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$$\frac{d}{dt}\left[V\right] = \frac{d}{dt}\left[80h^2\right]$$

$$40 = 160h\,\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{40}{160h} = \frac{1}{4h} \text{ ft/min.}$$

Answer:
When $h = 3$, water level is increasing at a rate of

$$\frac{dh}{dt} = \frac{1}{4\cdot 3} = \frac{1}{12} \text{ ft/min}$$