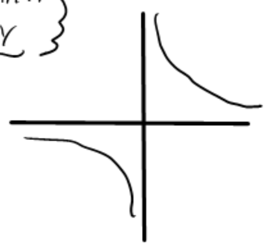
Section 2.5 Limits at Infinity

Goal Make sense out of limits like $\lim\limits_{x\to\infty} f(x)$

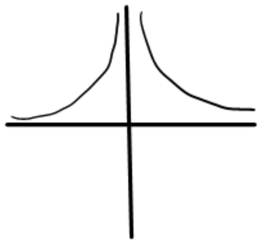
We start with a few examples that will guide the way.

$\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ (called a limit at infinity)

$\lim\limits_{x\to-\infty} \frac{1}{x^n} = 0$



$y = \frac{1}{x^n}$ (n odd)

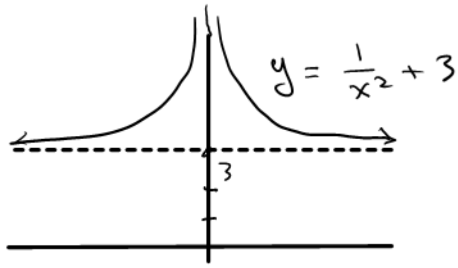


$y = \frac{1}{x^n}$ (n even)

In either case $\frac{1}{x^n}$ gets closer and closer to zero the bigger x gets (in either the positive or negative direction) and we express this condition by saying $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x^n} = 0$.

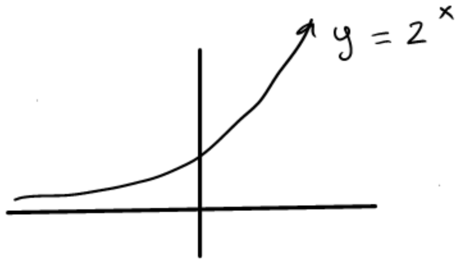
Example $\lim\limits_{x\to\infty} \left(\frac{1}{x^2} + 3\right) = 3$

$\lim\limits_{x\to-\infty} \left(\frac{1}{x^2} + 3\right) = 3$

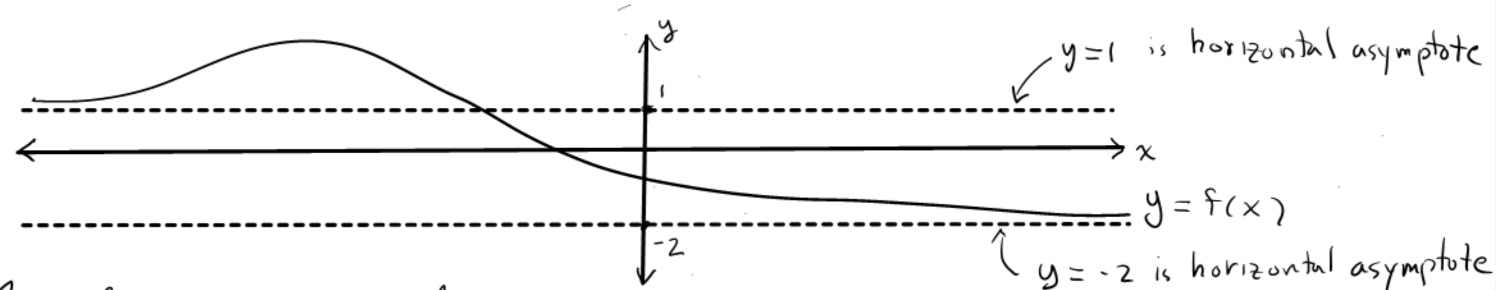


Example $\lim\limits_{x\to\infty} 2^x = \infty$

$\lim\limits_{x\to-\infty} 2^x = 0$



Definition If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then the line $y = L$ is called a horizontal asymptote of $f(x)$



$\lim\limits_{x\to\infty} f(x) = -2$ so line $y = -2$ is H.A.

$\lim\limits_{x\to-\infty} f(x) = 1$ so line $y = 1$ is H.A.

Note A function can have at most two horizontal asymptotes – one for $x \to \infty$ and one for $x \to -\infty$.

If $\lim\limits_{x\to\infty} f(x) = \pm\infty$, this is not a H.A.

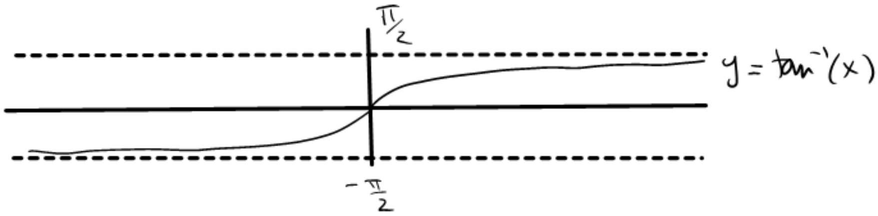
End Behavior of Functions

The "*End Behavior*" of a function is how it behaves at the "ends" of its domains, usually ∞ or $-\infty$. Asymptoes – both vertical and horizontal – carry information about end behavior.

End behavior of $\tan^{-1}(x)$

$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$$



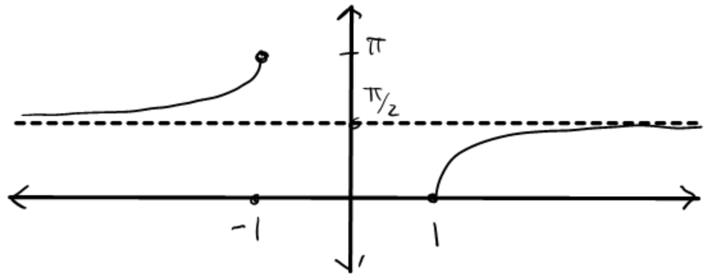
End behavior of $\sec^{-1}(x)$

$$\lim_{x \to \infty} \sec^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \sec^{-1}(x) = \frac{\pi}{2}$$

$$\lim_{x \to 1^+} \sec^{-1}(x) = 0$$

$$\lim_{x \to -1^-} \sec^{-1}(x) = \pi$$



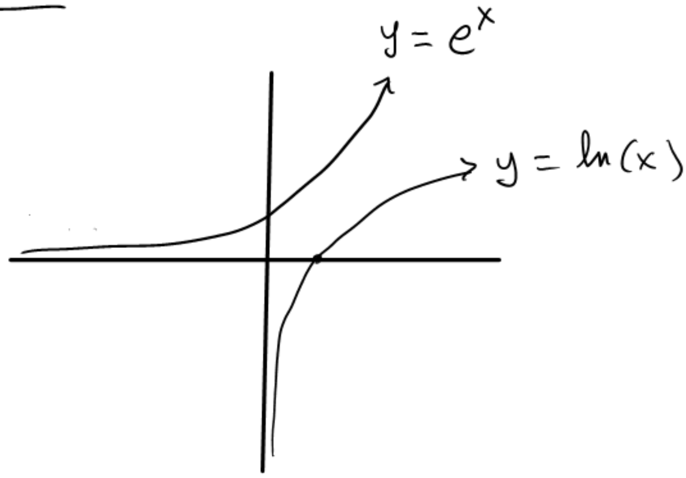
End behavior of e^x and $\ln(x)$

$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to \infty} \ln(x) = \infty$$

$$\lim_{x \to 0^+} \ln(x) = -\infty$$



End behavior of $\frac{1}{x^n}$

$$\lim_{x \to \infty} \frac{1}{x^n} = 0$$

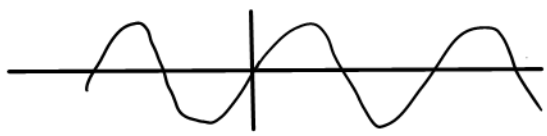
$$\lim_{x \to -\infty} \frac{1}{x^n} = 0$$

$$\lim_{x \to 0^+} \frac{1}{x^n} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x^n} = \begin{cases} \infty & \text{if } n \text{ even} \\ -\infty & \text{if } n \text{ odd} \end{cases}$$

Not all functions have such well-behaved end-behavior.

$$\lim_{x \to \infty} \sin(x) \text{ DNE}$$

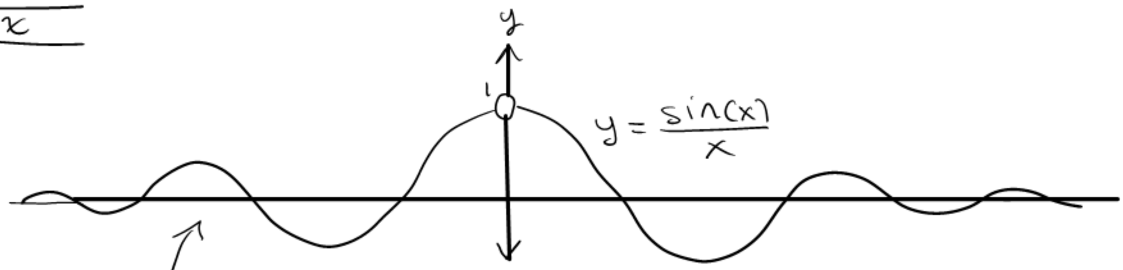


End behavior of $\frac{\sin(x)}{x}$

$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$

$\lim_{x\to-\infty} \frac{\sin(x)}{x} = 0$

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

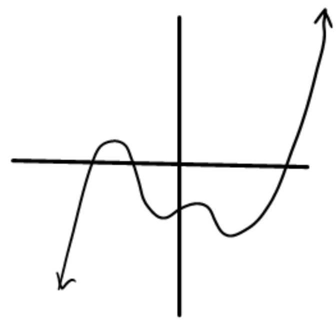


x-axis is a horizontal asymptote.

End Behavior of polynomials $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

$$\lim_{x\to\infty} p(x) = \begin{cases} \infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases}$$

$$\lim_{x\to-\infty} p(x) = \begin{cases} \infty & \text{if } n \text{ even } a_n > 0 \\ \infty & \text{if } n \text{ odd } a_n < 0 \\ -\infty & \text{if } n \text{ even } a_n < 0 \\ -\infty & \text{if } n \text{ odd } a_n > 0 \end{cases}$$



End Behavior of Rational Functions

This can get slightly tricky. Below we show a techinque for evaluating limits at ∞ of rational functions. In each case we multiply by 1 in the form $\frac{1/x^n}{1/x^n}$ where n is the highest power of x in the function.

Ex $\lim_{x\to\infty} \frac{5x^2+1}{2x^2+x} = \lim_{x\to\infty} \frac{5x^2+1}{2x^2+x}\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{5+\frac{1}{x^2}}{2+\frac{1}{x}} = \frac{5+0}{2+0} = \frac{5}{2}$ (line $y = 5/2$ is H.A.)

Ex $\lim_{x\to\infty} \frac{5x+1}{2x^2+x} = \lim_{x\to\infty} \frac{5x+1}{2x^2+x}\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\frac{5}{x}+\frac{1}{x^2}}{2+\frac{1}{x}} = \frac{0+0}{2+0} = 0$ (line $y=0$ is H.A.)

Ex $\lim_{x\to\infty} \frac{5x^2+1}{2x+1} = \lim_{x\to\infty} \frac{5x^2+1}{2x+1}\frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\overbrace{5+\frac{1}{x^2}}^{\text{approaches 5}}}{\underbrace{\frac{2}{x}+\frac{1}{x^2}}_{\text{Approaches 0 pos.}}} = \infty$ (No H.A.)

After awhile you learn to compare degrees on top and bottom and draw a conclusion based on That.

Ex $\lim_{x\to\infty} \frac{2x^5-4x^3+3x+1}{\sqrt{3}x^5+3x-7} = \frac{2}{\sqrt{3}}$ (highest degrees on top and bottom match. Answer is fraction of their coefficients)

Ex $\lim_{x\to\infty} \frac{2x^5-4x^3+3x+1}{4x^6} = 0$ (Answer 0 when highest power on bottom)

Ex $\lim_{x\to\infty} \frac{x^2+5x+1}{1-x} = -\infty$

$\lim_{x\to-\infty} \frac{x^2+5x+1}{1-x} = \infty$

} when highest power is on top answer is $\pm\infty$, with sign depending on situation – you have to analyze signs of top and bottom.

But don't forget techniques on previous page work only for limits of rational functions with $x \to \infty$. Other situations require their own techniques.

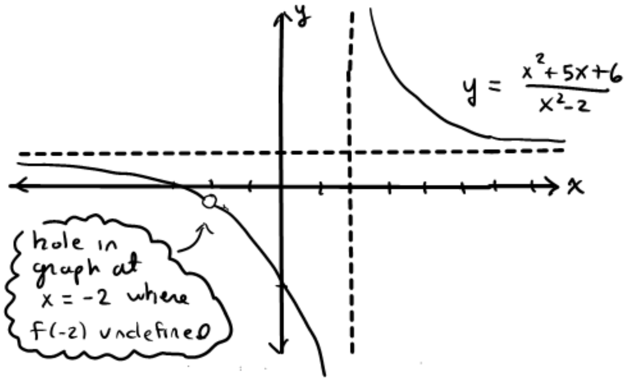
Ex $\displaystyle \lim_{x\to 1} \frac{x^2+5x+6}{x^2-4} = \frac{1^2+5\cdot 1+6}{1^2-4} = \frac{12}{-3} = \boxed{-4}$

Ex $\displaystyle \lim_{x\to -2} \frac{x^2+5x+6}{x^2-4} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-2)} = \lim_{x\to -2} \frac{x+3}{x-2} = \frac{-2+3}{-2-2} = \boxed{-\frac{1}{4}}$

Ex $\displaystyle \lim_{x\to 2^+} \frac{x^2+5x+6}{x^2-4} = \lim_{x\to 2^+} \frac{x+3}{x-2} = \boxed{\infty}$

Ex $\displaystyle \lim_{x\to \infty} \frac{x^2+5x+6}{x^2-4} = \lim_{x\to\infty} \frac{(x^2+5x+6)\,1/x^2}{(x^2-2)\,1/x^2} = \lim_{x\to\infty} \frac{1+\frac{5}{x}+\frac{6}{x^2}}{1-2/x^2} = \frac{1+0+0}{1-0} = \boxed{1}$

In more complex situations it is advisable to avoid short-cuts, which can sometimes lead to wrong answers if we are not careful.



Example

$$\lim_{x\to\infty} \frac{\sqrt{7x^2+4}}{3-5x} = \lim_{x\to\infty} \frac{\sqrt{7x^2+4}}{3-5x}\,\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{(7x^2+4)\frac{1}{x^2}}}{\frac{3}{x}-5} = \lim_{x\to\infty} \frac{\sqrt{7+\frac{4}{x^2}}}{\frac{3}{x}-5} = \frac{\sqrt{7+0}}{0-5} = \boxed{-\frac{\sqrt{7}}{5}}$$

$$\lim_{x\to -\infty} \frac{\sqrt{7x^2+4}}{3-5x} = \lim_{x\to -\infty} \frac{\sqrt{7x^2+4}}{3-5x}\,\frac{-1/x}{-1/x} = \lim_{x\to -\infty} \frac{\sqrt{(7x^2+4)(-\frac{1}{x})^2}}{-\frac{3}{x}+5} = \lim_{x\to -\infty} \frac{\sqrt{7+\frac{4}{x^2}}}{-\frac{3}{x}+5} = \frac{\sqrt{7+0}}{0+5} = \boxed{\frac{\sqrt{7}}{5}}$$

Here the lines $y = -\frac{\sqrt{7}}{5}$ and $y = \frac{\sqrt{7}}{5}$ are horizontal asymptotes.

Remember: $\sqrt{a}\; b = \sqrt{ab^2}$ only if b is positive
Example $\sqrt{4}(-3) \neq \sqrt{4(-3)^2}$
Since $x \to -\infty$ $1/x$ is negative – we had to change it to $-1/x$ to make it positive.

Ex $\displaystyle \lim_{x\to\infty} \left(\sqrt{x+4} - \sqrt{x+1}\right) = \lim_{x\to\infty} \left(\sqrt{x+4} - \sqrt{x+1}\right) \frac{\sqrt{x+4}+\sqrt{x+1}}{\sqrt{x+4}+\sqrt{x+1}}$

$\displaystyle = \lim_{x\to\infty} \frac{x+4-(x+1)}{\sqrt{x+4}+\sqrt{x+1}} = \lim_{x\to\infty} \frac{3}{\sqrt{x+4}+\sqrt{x+1}} = \boxed{0}$

Ex $\displaystyle \lim_{x\to\infty} \ln\left(\frac{x^2+1}{3x^2+2}\right) = \ln\left(\lim_{x\to\infty} \frac{x^2+1}{3x^2+2}\right) = \ln\left(\frac{1}{3}\right)$

[You can ignore the material on slant asymptotes]