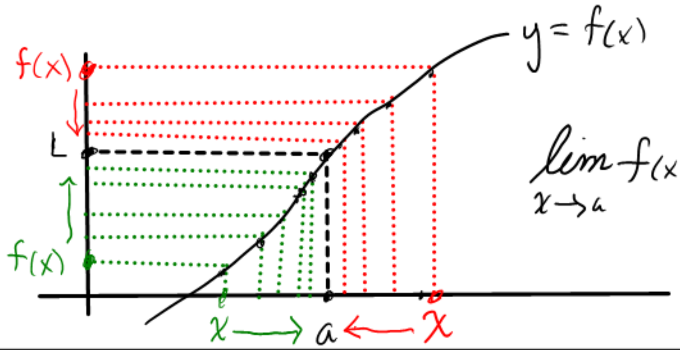
Section 2.3 Techniques for Computing Limits (Continued)

Let's begin by recalling important information from last time we met.

Recall

Given a function $f(x)$ and a number a, $\lim_{x\to a} f(x)$ is the number L that the value $f(x)$ approaches as x approaches a



$$\lim_{x\to a} f(x) = L$$

Theorem 2.3 (Limit Laws) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

1. $\lim_{x\to a}\left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}\left(f(x)-g(x)\right) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} c\,f(x) = c\left(\lim_{x\to a} f(x)\right)$
4. $\lim_{x\to a}\left(f(x)\,g(x)\right) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (provided $\lim_{x\to a} g(x) \neq 0$)
6. $\lim_{x\to a}\left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n$
7. $\lim_{x\to a}\left(f(x)\right)^{\frac{n}{m}} = \left(\lim_{x\to a} f(x)\right)^{\frac{n}{m}}$ (But if m even we require $f(x)\geq 0$ when x near a)

Sometimes, like if $f(x)$ is a polynomial or rational function whose domain includes a, the limit laws produce

$$\lim_{x\to a} f(x) = f(a),$$

that is, to compute the limit just plug in a to $f(x)$.

But more often we'll have a limit of form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where $g(a)=0$. In such a case we can't just plug in a. We algebraically cancel the term that makes $g(a)=0$ and <u>then</u> apply the relevant limit laws.

Example

(denominator $2-x$ approaches 0 as $x \to 2$ Thus we will try to cancel it)

$$\lim_{x\to 2} \frac{x^4+x^3-6x^2}{2-x} = \lim_{x\to 2} \frac{x^2(x^2+x-6)}{2-x} = \lim_{x\to 2} \frac{x^2(x+3)(x-2)}{2-x}$$

$$= \lim_{x\to 2} \frac{x^2(x+3)\cancel{(x-2)}}{-\cancel{(x-2)}} = \lim_{x\to 2} -x^2(x+3) = -2^2(2+3) = \boxed{-20}$$

Example

(Denominator approaches zero, so try to cancel it)

$$\lim_{h\to 0} \frac{\sqrt{5+h}-\sqrt{5}}{h} = \lim_{h\to 0} \frac{\sqrt{5+h}-\sqrt{5}}{h} \cdot \frac{\sqrt{5+h}+\sqrt{5}}{\sqrt{5+h}+\sqrt{5}}$$

$$= \lim_{h\to 0} \frac{\sqrt{5+h}^2+\sqrt{5+h}\sqrt{5}-\sqrt{5}\sqrt{5+h}-\sqrt{5}^2}{h(\sqrt{5+h}+\sqrt{5})} = \lim_{h\to 0} \frac{5+h-5}{h(\sqrt{5+h}+\sqrt{5})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{5+h}+\sqrt{5})} = \lim_{h\to 0} \frac{1}{\sqrt{5+h}+\sqrt{5}} = \frac{1}{\sqrt{5+0}+\sqrt{5}} = \boxed{\frac{1}{2\sqrt{5}}}$$

Example

$$\lim_{x\to -3} \frac{x+3}{x^2+4x+3} = \lim_{x\to -3} \frac{\cancel{x+3}}{\cancel{(x+3)}(x+1)} = \lim_{x\to -3} \frac{1}{x+1} = \frac{1}{-3+1} = \boxed{-\frac{1}{2}}$$

Notes

Never write this: $\lim_{x\to -3} \frac{x+3}{(x+3)(x-1)} = \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$

not equal! -1 not equal. -1

Continue writing $\lim_{x\to a}$ until reaching the point at which the a is plugged in.

Likewise, don't write the final step as, say, $\lim_{x\to -3} = -\frac{1}{2}$

Syntax is $\lim_{x\to a} f(x) = -\frac{1}{2}$.

One-Sided Limits

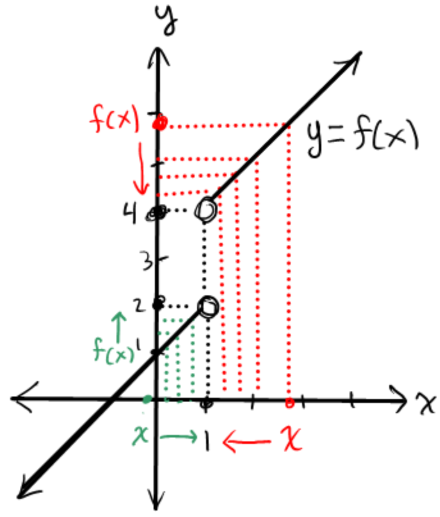
Recall that to have $\lim_{x \to a} f(x) = L$ it must be the case that $f(x)$ approaches L no matter how x "gets to" the number a. Here is an example where $\lim_{x \to a} f(x)$ DNE but we can still make sense of the limit as x approaches a either from the left (or right)

Example

$$f(x) = x + \frac{|x-1|}{x-1} + 2 = \begin{cases} x+1 & \text{if } x<1 \\ x+3 & \text{if } x>1 \end{cases}$$

If $x<1$, then $x-1$ is negative, so $|x-1| = -(x-1)$ and hence $\frac{|x-1|}{x-1} = -1$.

But if $x>1$ then $x-1$ is positive, so $|x-1| = x-1$ making $\frac{|x-1|}{x-1} = 1$



$\lim_{x \to 1} f(x)$ DNE (because $f(x)$ does not approach a single value as x approaches 1)

$\lim_{x \to 1^+} f(x) = 4$ ← Right-hand limit means x approaches $a=1$ from the right

$\lim_{x \to 1^-} f(x) = 2$ ← Left-hand limit means x approaches $a=1$ from the left

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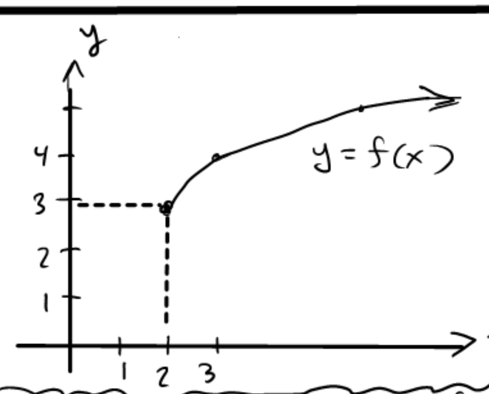
Example

$f(x) = \sqrt{x-2} + 3$

$\lim_{x \to 2^+} f(x) = 3$

$\lim_{x \to 2^-} f(x)$ DNE (function not even defined for x to left of 2)

$\lim_{x \to 2} f(x)$ DNE



graph of $f(x)$ is graph of $y=\sqrt{x}$ shifted right 2 units and up 3 units

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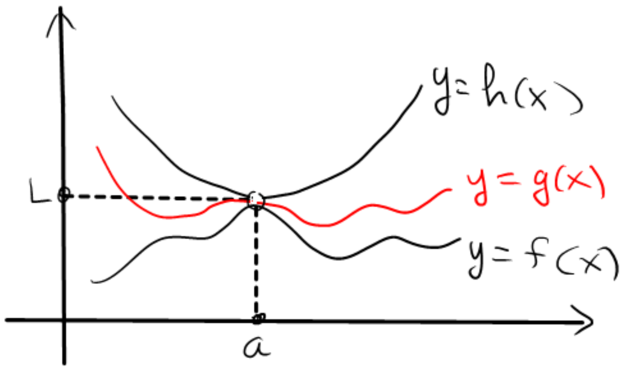
Theorem 2.1 $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$

Trigonometric Limits

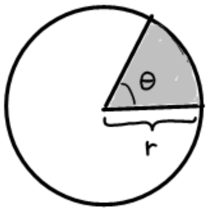
Now let's look at limits involving trigonometric functions.
Our main goal is to find $\lim_{x \to 0} \frac{\sin(x)}{x}$, which will be useful later.
We will need the following two ideas:

Theorem 2.5 (Squeeze Theorem) For finding $\lim_{x \to a} g(x)$

Suppose f, g, and h are three functions for which $f(x) \leq g(x) \leq h(x)$ for all x near a, except possibly at a (where they may be undefined). If $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$, then also $\lim_{x \to a} g(x) = L$.



Area of a sector



The area of the sector of a circle of radius r and radian measure θ is $A = \frac{r^2\theta}{2}$

Reason:
$$A = (\pi r^2)\left(\frac{\theta}{2\pi}\right) = \frac{r^2\theta}{2}$$

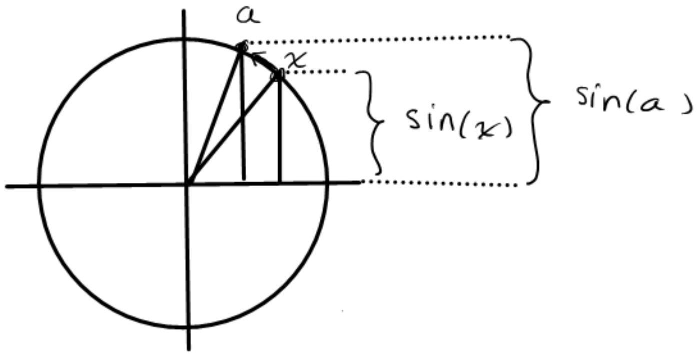
area of entire circle

fraction of 2π radians

Limits of sin and cos

Fact: $\lim_{x \to a} \sin(x) = \sin(a)$

also $\lim_{x \to a} \cos(x) = \cos(a)$



Consequently also $\lim_{x \to a} \tan(x) = \tan(a)$ (If a is in domain of tan)

etc.

Examples

$$\lim_{x \to \pi} \cos(x) = \cos(\pi) = -1$$

$$\lim_{x \to \pi} \sin(x) = \sin(\pi) = 0$$

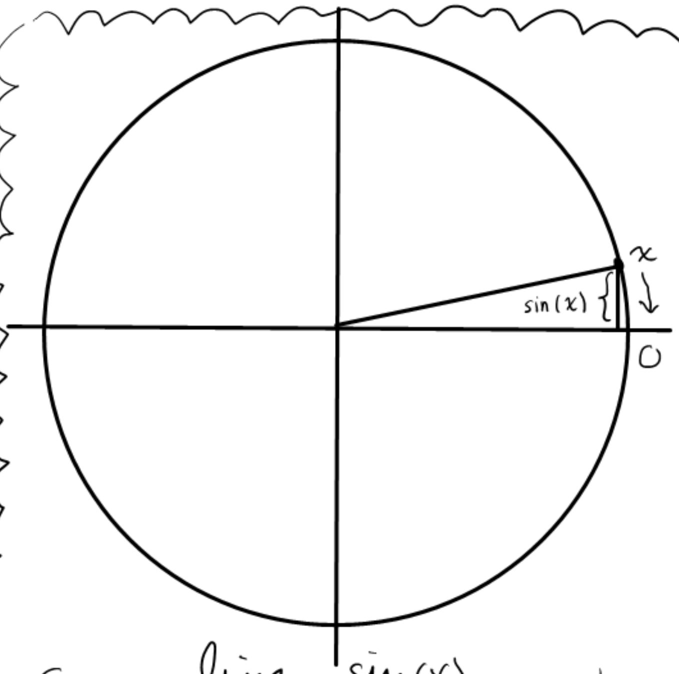
$$\lim_{x \to 0} \cos^2(x) = \left(\lim_{x \to 0} \cos(x)\right)^2 = (\cos(0))^2 = 1^2 = 1$$

Problem Find $\lim_{x \to 0} \frac{\sin(x)}{x}$

The limit is tricky because we can't cancel the x in the denominator (which approaches 0). To overcome this we use the squeeze Theorem. We seek the following set-up:

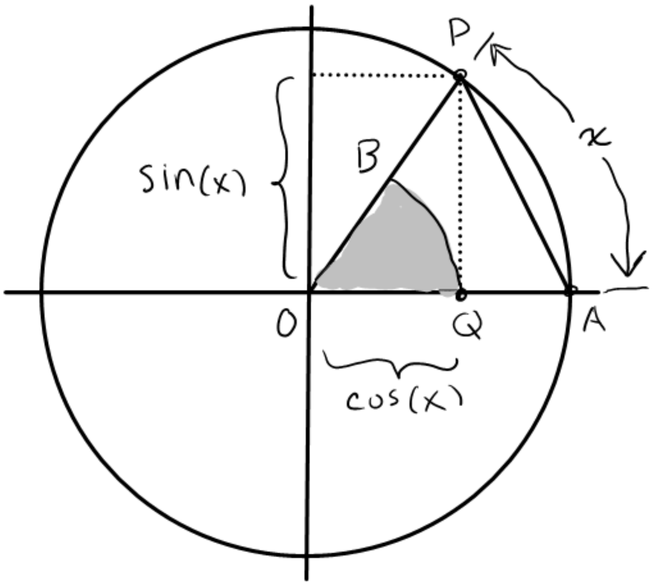
$$f(x) \le \frac{\sin(x)}{x} \le h(x)$$

should be simple functions with $\lim_{x\to 0} f(x) = \lim_{x \to 0} h(x)$



Guess: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ because for small x, $\sin(x) \approx x$.

Consider this picture:



$$\text{Area}\begin{pmatrix}\text{sector}\\ OQB\end{pmatrix} \le \text{Area}\begin{pmatrix}\text{triangle}\\ OAP\end{pmatrix} \le \text{Area}\begin{pmatrix}\text{sector}\\ OAP\end{pmatrix}$$

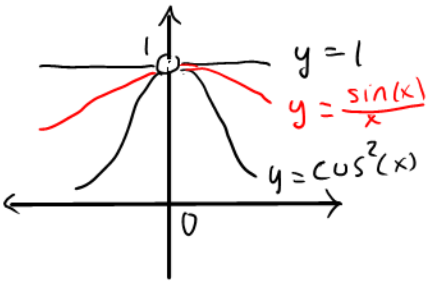
$$\frac{\cos^2(x)|x|}{2} \le \frac{1}{2}(1)\left|\sin(x)\right| \le \frac{1^2|x|}{2}$$

$$\cos^2(x)\,|x| \le |\sin(x)| \le |x|$$

$$\cos^2(x) \le \frac{|\sin(x)|}{|x|} \le 1$$

$$\cos^2(x) \le \frac{\sin(x)}{x} \le 1$$

$\lim_{x\to 0} \cos^2(x) = 1 = \lim_{x\to 0} 1$



Thus by Squeeze Theorem $\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$

Exercise Show that if we used degrees (not radians) then $\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{\pi}{180}$ (not 1)

This is why we use radians, and the unit circle as a natural protractor. They make equations like these work out neatly.

Remember $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$. It will pop up again.

Knowing this limit helps us with others:

<u>Example</u>

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = \lim_{x\to 0} \frac{\cos(x)-1}{x}\,\frac{\cos(x)+1}{\cos(x)+1}$$

$$= \lim_{x\to 0} \frac{\cos^2(x)-1}{x(\cos(x)+1)}$$

$$= \lim_{x\to 0} \frac{-\sin^2(x)}{x(\cos(x))}$$

$$= \lim_{x\to 0} \frac{\sin(x)}{x}\,\frac{-\sin(x)}{\cos(x)+1}$$

$$= \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x\to 0} \frac{-\sin(x)}{\cos(x)+1}\right)$$

$$= 1 \cdot \frac{-\sin(0)}{\cos(0)+1}$$

$$= 1 \cdot \frac{0}{1+1} = \boxed{0}$$

Remember

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

<u>Example</u>

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\frac{\sin(x)}{\cos(x)}}{\frac{x}{1}} = \lim_{x\to 0} \frac{\sin(x)}{x}\,\frac{1}{\cos(x)}$$

$$= 1 \cdot \frac{1}{1} = \boxed{1}$$