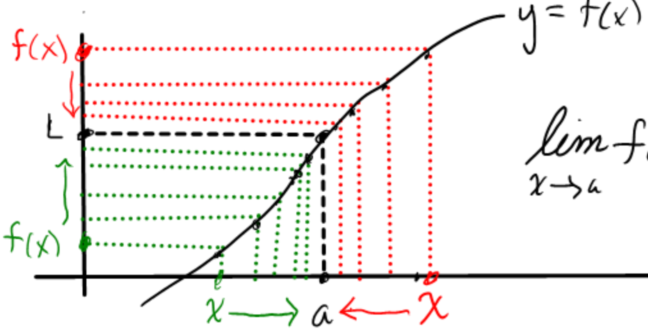
Section 2.3 Techniques for Computing limits

Last time we introduced the mathematical concept of a limit because it can be used to compute slopes of tangent lines. Today we will forget about tangent lines for the moment and concentrate on computing limits. Once we have limits under control (Part II of text) we'll return to tangent slopes in Part III

Recall

Given a function $f(x)$ and a number a, $\lim\limits_{x\to a} f(x)$ is the number L that the value $f(x)$ approaches as x approaches a



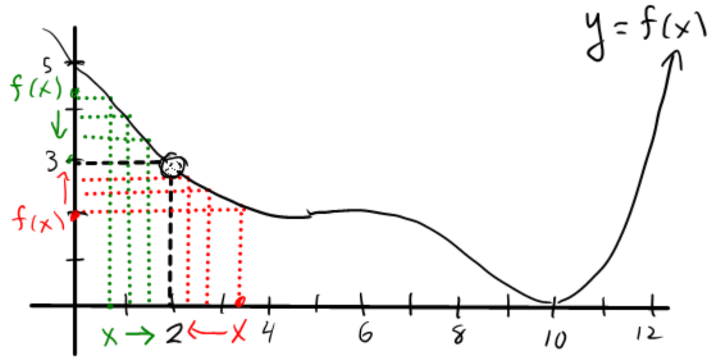
$$\lim_{x\to a} f(x) = L$$

Example

$$\lim_{x\to 2} f(x) = 3$$

$$\lim_{x\to 0} f(x) = 5$$

$$\lim_{x\to 10} f(x) = 0$$



Note that in the above example, even though $f(2)$ is not defined, we still have $\lim\limits_{x\to 2} f(x) = 3$. This is an important point. Limits can give information about how $f(x)$ behaves near a "bad value" of x.

Example

Last time we worked out $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 2$

Here $x=1$ is a "bad value" for the function $f(x)=\dfrac{x^2-1}{x-1}$ because $f(1)$ is not defined. (ie 1 is not in the domain of $f(x)$).

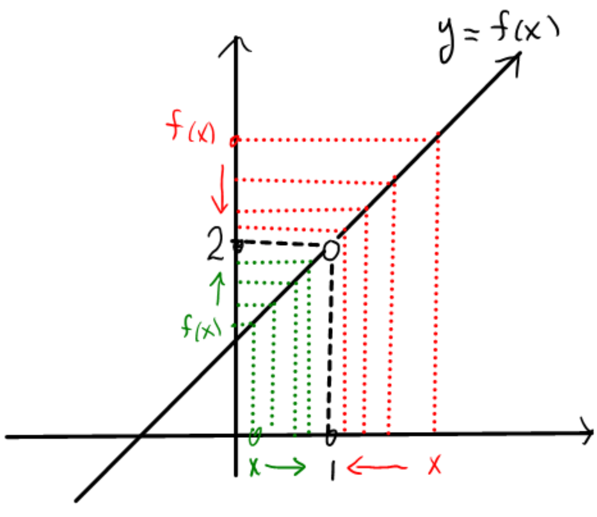
But even though we can't do $f(1)$ we do have $\lim\limits_{x\to 1} f(x) = 2$.

Let's look at this from the point of view of a graph.

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1}$$

$$= (x+1)\frac{x-1}{x-1} = x+1$$

provided $x \neq 1$

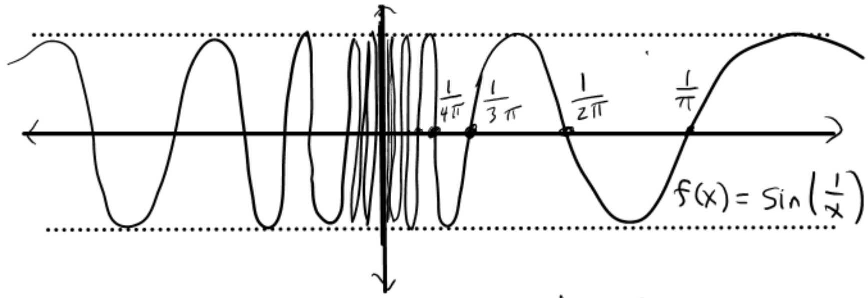


Thus $f(x) = x+1$ so its graph is a line with slope 1 and y-intercept 1, except $f(1)$ not defined. (So hole in graph at point where $x=1$.) But $\lim\limits_{x\to 1} f(x) = 2$.

Warning: Some limits make no sense. If so we say they do not exist (DNE)

Example $f(x) = \sin\left(\frac{1}{x}\right)$

$f(0)$ is not defined, but can we do $\lim\limits_{x\to 0} f(x)$?



As $x \to 0$, $\frac{1}{x} \to \infty$, and so our function $f(x) = \sin\left(\frac{1}{x}\right)$ just oscilates up and down, approaching no one value L

Conclusion $\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$ DNE (does not exist)

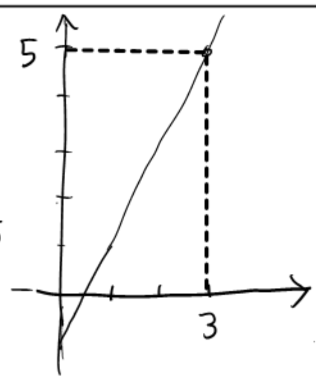
The text introduces certain Limit Laws that help us compute limits.

Theorem 2.2

If $f(x) = mx + b$ then $\lim\limits_{x\to a} f(x) = f(a)$ (i.e. to do The limit just plug a into f(x))

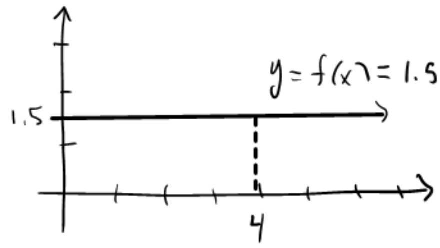
Example

$$\lim_{x\to 3} (2x-1) = 2\cdot 3 - 1 = 5$$



Example ($m = 0$, $b = 1.5$)

$$\lim_{x\to 4} 1.5 = 1.5$$



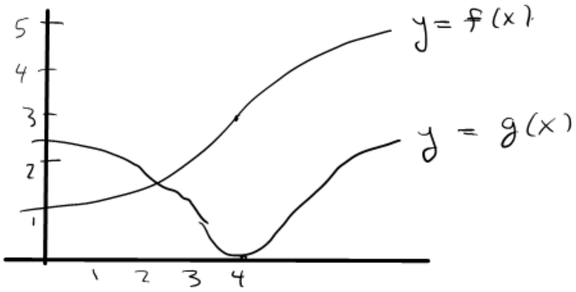
For more complex limits we can use the following limit laws.

Theorem 2.3 (Limit Laws) If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist, then

1. $\lim\limits_{x\to a} (f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
2. $\lim\limits_{x\to a} (f(x) - g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a} c\, f(x) = c\left(\lim\limits_{x\to a} f(x)\right)$
4. $\lim\limits_{x\to a} (f(x)\, g(x)) = \left(\lim\limits_{x\to a} f(x)\right)\cdot\left(\lim\limits_{x\to a} g(x)\right)$
5. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ (provided $\lim\limits_{x\to a} g(x) \neq 0$)
6. $\lim\limits_{x\to a} (f(x))^n = \left(\lim\limits_{x\to a} f(x)\right)^n$
7. $\lim\limits_{x\to a} (f(x))^{\frac{n}{m}} = \left(\lim\limits_{x\to a} f(x)\right)^{\frac{n}{m}}$ (But if m even we require $f(x) \geq 0$ when x near a)

Example

$$\lim_{x\to 4}\sqrt{\frac{5f(x)}{g(x)+3}} = \sqrt{\lim_{x\to 4}\frac{5f(x)}{g(x)+3}}$$



$$= \sqrt{\frac{\lim_{x\to 4} 5f(x)}{\lim_{x\to 4}(g(x)+3)}} = \sqrt{\frac{5\cdot\lim_{x\to 4} f(x)}{\lim_{x\to 4} g(x) + \lim_{x\to 4} 3}}$$

$$= \sqrt{\frac{5\cdot 3}{0+3}} = \sqrt{\frac{15}{3}} = \boxed{\sqrt{5}}$$ (ok to skip steps on problems like these!)

Example $\lim_{x\to -3}\left(4x^2-2x+3\right) = 4(-3)^2-2(-3)+4 = 36+6+4 = \boxed{46}$

Example $\lim_{x\to 2}\frac{x^2+1}{x-4} = \ldots$ (limit laws) $\ldots = \frac{2^2+1}{2-4} = \boxed{-\frac{5}{2}}$

Thinking like this leads to

> **Theorem 2.4**
>
> If $p(x)$ is a polynomial, $\lim_{x\to a} p(x) = p(a)$
>
> If $\frac{p(x)}{q(x)}$ is a rational function, $\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ (provided $q(a)\neq 0$)

BUT The most significant limits for us will have form $\lim_{x\to a}\frac{p(x)}{q(x)}$ with $q(a)=0$, so limit laws will not immediately apply.

For such limits our strategy is to first algebraicly cancel the term that makes $g(a)=0$, and then apply the limit laws

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Example

cant just plug in $x=0$ here because that creates division by zero

$$\lim_{x\to 0}\frac{5x^3+3x^2+7x}{x}$$

$$= \lim_{x\to 0}\frac{\cancel{x}(5x^2+3x+7)}{\cancel{x}} = \lim_{x\to 0}(5x^2+3x+7) = \boxed{0}$$

Example

$$\lim_{x\to 0}\frac{x^2-1}{x-1} = \lim_{x\to 0}\frac{(x+1)(x-1)}{x-1} = \lim_{x\to 0}(x+1) = 1+1 = \boxed{2}$$

Example

Denominator $x-2$ approaches 0, so can't just plug in $x=2$.
Strategy: Combine fractions on top & see if anything cancels.

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=\lim_{x\to 2}\frac{\frac{2}{2}\frac{1}{x}-\frac{1}{2}\frac{x}{x}}{x-2}=\lim_{x\to 2}\frac{\frac{2}{2x}-\frac{x}{2x}}{x-2}$$

$$=\lim_{x\to 2}\frac{\frac{2-x}{2x}}{\frac{x-2}{1}}=\lim_{x\to 2}\frac{2-x}{2x}\,\frac{1}{x-2}=\lim_{x\to 2}\frac{-\cancel{(x-2)}\cdot 1}{2x\cancel{(x-2)}}$$

$$=\lim_{x\to 2}\frac{-1}{2x}=-\frac{1}{2\cdot 2}=\boxed{-\frac{1}{4}}$$

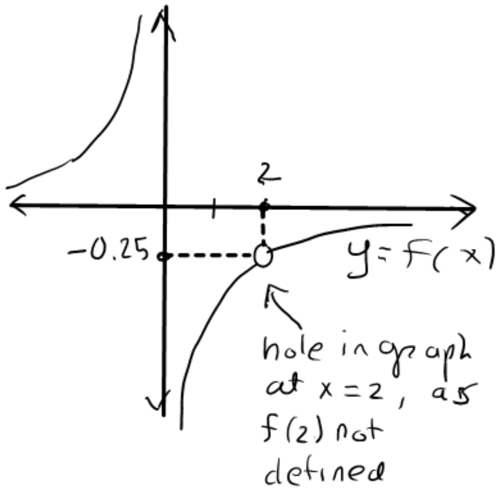
Let's also look at this problem numerically and graphically to see exactly what's going on. We have

$$f(x)=\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=\frac{\frac{2-x}{2x}}{\frac{x-2}{1}}=-\frac{1}{2x}\,\frac{x-2}{x-2}\overset{\text{provided } x\neq 2}{=}-\frac{1}{2x}$$

x	$f(x)=\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=-\frac{1}{2x}$
2.1	-0.23809
2.01	-0.24875
2.001	-0.24987
2.0001	-0.24999
↓	↓
2	$-0.25=-\frac{1}{4}$

Note Table supports our answer of $\lim_{x\to 2} f(x)=-\frac{1}{4}$

We can also draw the graph of $f(x)=\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=-\frac{1}{x}$. It's the graph of $y=\frac{1}{x}$ scaled by a factor of $\frac{1}{2}$ and reflected across the x-axis



Idea: $\lim_{x\to 2} f(x)=-\frac{1}{4}$ tells us how $f(x)$ behaves near "bad point" $x=2$

Example $\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=\frac{\frac{1}{3}-\frac{1}{2}}{3-2}=\frac{1}{3}-\frac{1}{2}=\frac{2}{2}\frac{1}{3}-\frac{1}{2}\frac{3}{3}=\frac{2}{6}-\frac{3}{6}=\boxed{-\frac{1}{6}}$

(Easier, because $x=3$ not a "bad point" for function $f(x)=\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$)

Example $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

Plugging in $x=4$ gives $\frac{0}{0}$, which is not defined.
Thus try to cancel the denominator.
Use trick of multplying by conjugate over itself.

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\,\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to 4}\frac{\sqrt{x}^2+2\sqrt{x}-2\sqrt{x}-4}{(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=$$

$$\frac{1}{\sqrt{4}+2}=\frac{1}{2+2}=\boxed{\frac{1}{4}}$$

Alternate approach $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{\cancel{\sqrt{x}-2}}{(\sqrt{x}+2)\cancel{(\sqrt{x}-2)}}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{\sqrt{4}+2}=\boxed{\frac{1}{4}}$

{difference of two squares. $x-4=\sqrt{x}^2-2^2$}