Section 5.4 The Fundamental Theorem of Calculus (continued)

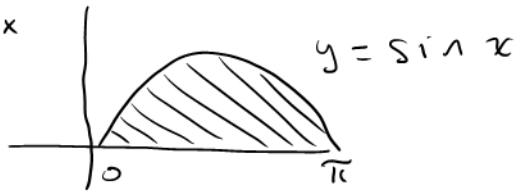
Recall

Suppose $f(x)$ is continuous on $[a,b]$. Then:

F.T.C. I $\quad \dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

F.T.C. II If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b)-F(a)$.

F.T.C. II is particularly useful because it gives a formula for the definite integral $\int_a^b f(x)\,dx$



Example Find the area:

Answer will be $\int_0^\pi \sin x\,dx$.

To find this value, FTC II says we need an antiderivative of $\sin x$
This is easy. The antiderivatives are $-\cos x + C$
But FTC says we can use any antiderivative, say $F(x) = -\cos x + 0$

Thus area is $\int_0^\pi \sin x\,dx = -\cos(\pi) - (-\cos(0)) = -(-1)+1 =$ **2 sq. units**

Notation Given a function $F(x)$, we define $F(b)-F(a) = F(x)\Big]_a^b = \big[F(x)\big]_a^b$

Thus F.T.C II says $\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b)-F(a)$

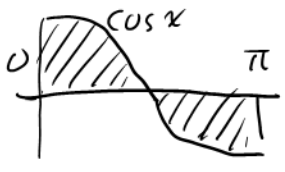
Ex $\displaystyle\int_1^e \frac{1}{x}\,dx = \ln|x|\Big]_1^e = \ln|e| - \ln|1| = 1-0 = \boxed{1}$

Ex $\displaystyle\int_2^3 x^3\,dx = \frac{x^4}{4}\Big]_2^3 = \frac{3^4}{4} - \frac{2^4}{4} = \frac{81}{4} - \frac{16}{4} = \boxed{\frac{65}{4}}$

Ex $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big]_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{4} - \frac{-\pi}{4} = \boxed{\frac{\pi}{2}}$

Ex $\displaystyle\int_0^\pi \cos x\,dx = \sin x\Big]_0^\pi = \sin\pi - \sin 0 = 0-0 = \boxed{0}$

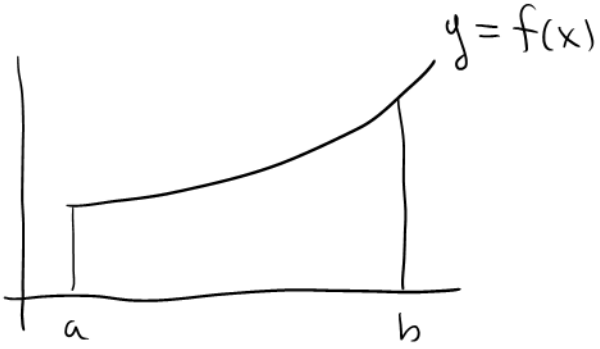
Note: Answer makes sense:
$\int_0^\pi \cos x\,dx = A_{up} - A_{down} = 0$



Application The average value of $f(x)$ on $[a,b]$

Question What is the average value of $f(x)$ on $[a,b]$?

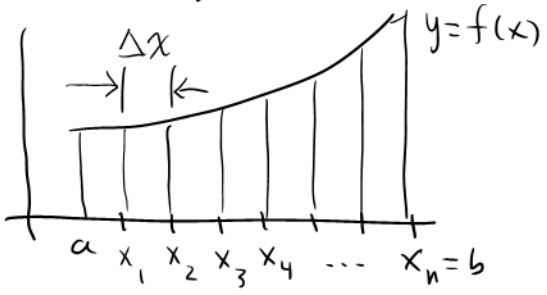
Under normal circumstances we would add up all the $f(x)$ values and divide by the number of values.



But this makes no sense because there are ∞ many $f(x)$ values.

To resolve this difficulty take x-values $a < x_1 < x_2 < x_3 \cdots < x_n = b$ as illustrated.

As usual, put $\Delta x = \dfrac{b-a}{n}$



Average value of $f(x)$ on $[a,b]$ is approximately

$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} = \sum_{k=1}^{n} f(x_k)\frac{1}{n} = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\frac{b-a}{n} = \boxed{\frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x}$$

To get the exact ave. value, let the number n of sample points go to ∞:

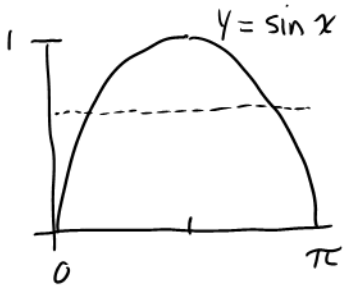
$$\left(\text{Ave. Value of } f(x) \text{ on } [a,b]\right) = \lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \frac{1}{b-a}\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Conclusion Average value of $f(x)$ on $[a,b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Ex What's the average value of $\sin x$ on $[0,\pi]$?

Guess: About 0.6?



Exact answer: $\dfrac{1}{\pi-0}\displaystyle\int_0^{\pi} \sin x\,dx$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{1}{\pi}\left(-\cos\pi - (-\cos 0)\right) = \frac{1}{\pi}(1+1) = \boxed{\frac{2}{\pi}} \approx 0.636619$$

Integrating Rates of Change

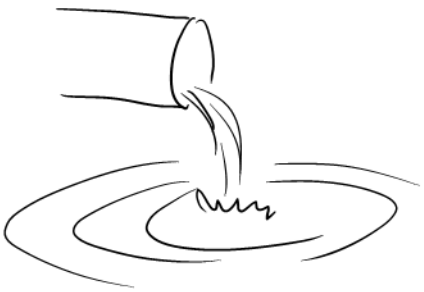
FTC II $\int_a^b F'(t)\,dt = F(b) - F(a)$

$F'(t)$: rate of change of $F(t)$

$F(b) - F(a)$: net change in quantity $F(t)$ between times $t=a$ and $t=b$

Thus $\int_a^b \left(\begin{array}{c}\text{rate of change}\\ \text{of a quantity}\end{array}\right) dt = \left(\begin{array}{c}\text{total change in}\\ \text{quantity between}\\ \text{times } a \text{ \& } b\end{array}\right)$

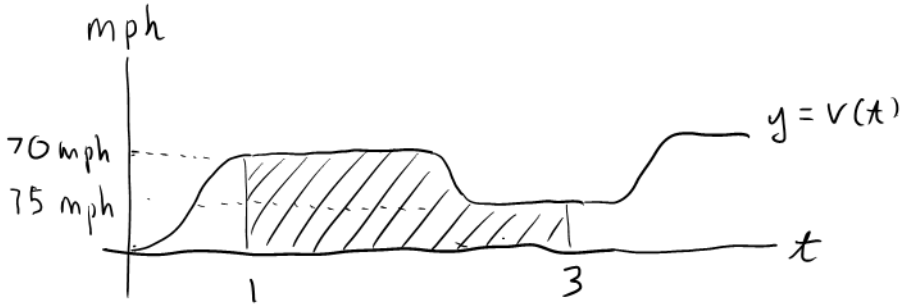
Ex



$f(t) = \left(\begin{array}{c}\text{gallons/sec of water flowing}\\ \text{out of a pipe at time } t\end{array}\right)$

$\int_1^{10} f(t)\,dt = \left(\begin{array}{c}\text{total \# of gallons that}\\ \text{pass through the pipe between}\\ 1{:}00 \text{ and } 10{:}00\end{array}\right)$

Ex $v(t)$ = velocity of an object at time t
= rate of change of position $s(t)$

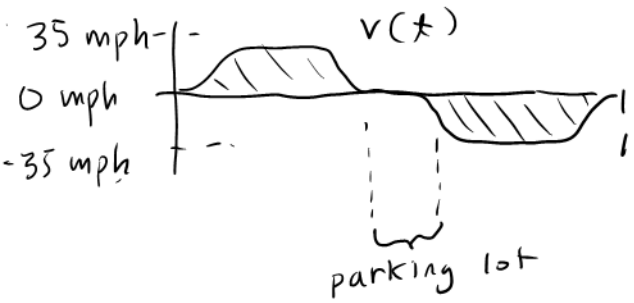


velocities at various times t during a car trip. - Sometimes going 35 mph, sometimes 70 mph etc

$\int_1^3 v(t)\,dt = s(3) - s(1) = \left(\begin{array}{c}\text{distance between cars'}\\ \text{locations at times } t=1\\ \text{and } t=3\end{array}\right)$

$= (\text{shaded area})$

Ex Drive to store & return home



$\int_0^1 v(t)\,dt = A_{up} - A_{down} = 0$

(0 because you returned to your starting point. Net change in position is 0.)