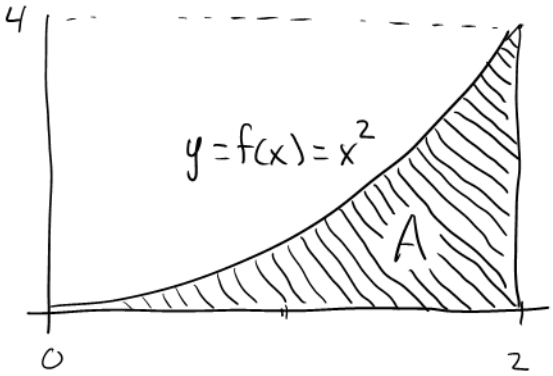
Sections 5.2 and 5.3 Sigma Notation and The Definite Integral

Now that we've developed sigma notation, lets use it to find the area under a curve.

Recall $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{k=1}^{n} c f(k) = c \sum_{k=1}^{n} f(k)$

Example

Find the area under $y = f(x) = x^2$ between $x = 0$ and $x = 2$

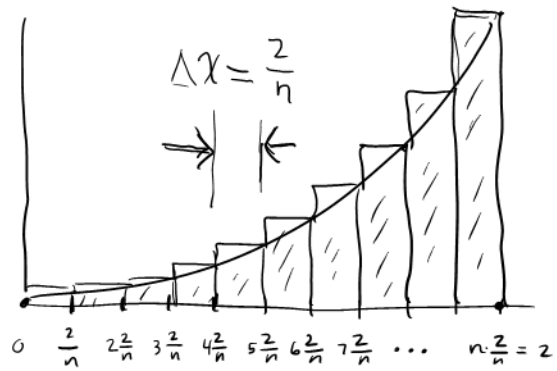


$A = ?$

Begin by dividing $[0, 2]$ into n pieces, each of length

$$\Delta x = \frac{2}{n}$$

On each interval, make a rectangle, as illustrated

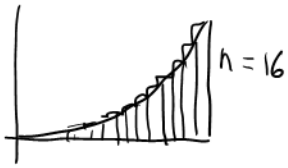


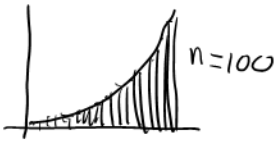
Area of rectangle #k is $\underbrace{f\left(k \cdot \frac{2}{n}\right)}_{\text{height}} \underbrace{\Delta x}_{\text{base}} = \left(\frac{2k}{n}\right)^2 \frac{2}{n} = \frac{8k^2}{n^3}$

$$\begin{pmatrix}\text{sum of areas of} \\ n \text{ rectangles}\end{pmatrix} = \sum_{k=1}^{n} f\left(k \cdot \frac{2}{n}\right) \Delta x = \sum_{k=1}^{n} \frac{8k^2}{n^3} =$$

$$= \frac{8}{n^3} \sum_{k=1}^{n} k^2 = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{4(n+1)(2n+1)}{3} = \boxed{\frac{4}{3} \frac{2n^2 + 3n + 1}{n^2}}$$

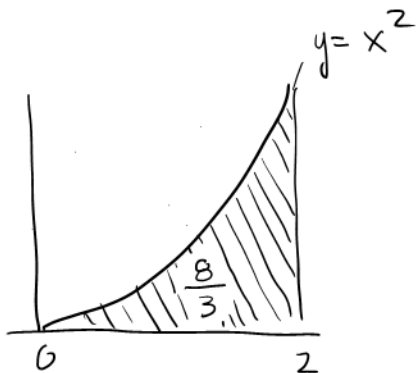
For better approx make # of rectangles n larger.





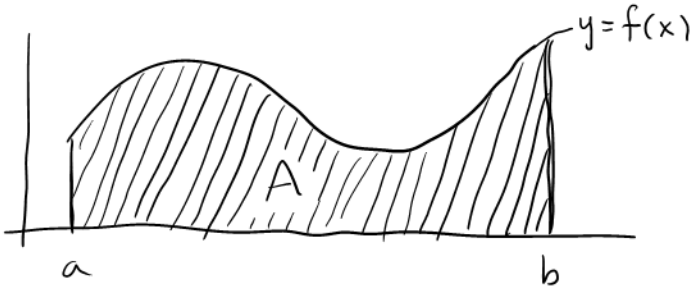
$$A = \lim_{n \to \infty} \begin{pmatrix}\text{sum of areas} \\ \text{of } n \text{ rectangles}\end{pmatrix} =$$

$$= \lim_{n \to \infty} \frac{4}{3} \frac{2n^2 + 3n + 1}{n^2} = \frac{4}{3} \cdot 2 = \boxed{\frac{8}{3} \text{ square units}}$$



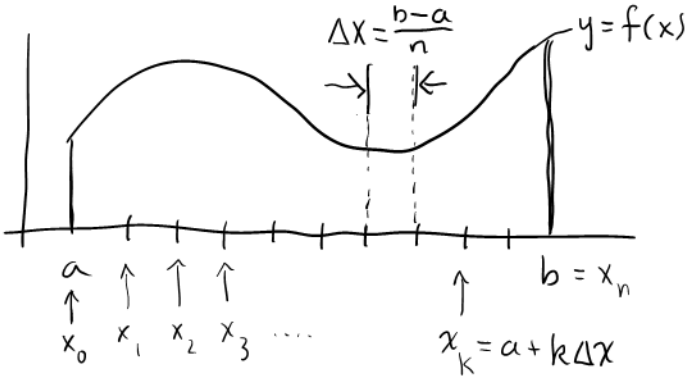
Summary

How to find the area of the region under $y = f(x)$ between a & b.

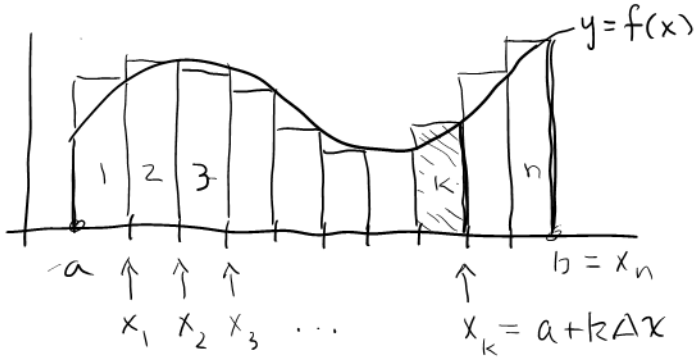


① Subdivide $[a,b]$ into n equal pieces, each of length $\Delta x = \frac{b-a}{n}$.

Get points $x_k = a + k\Delta x$ equally spaced on $[a,b]$



② On each subinterval $[x_{k-1}, x_k]$ establish a rectangle of height $f(x_k)$



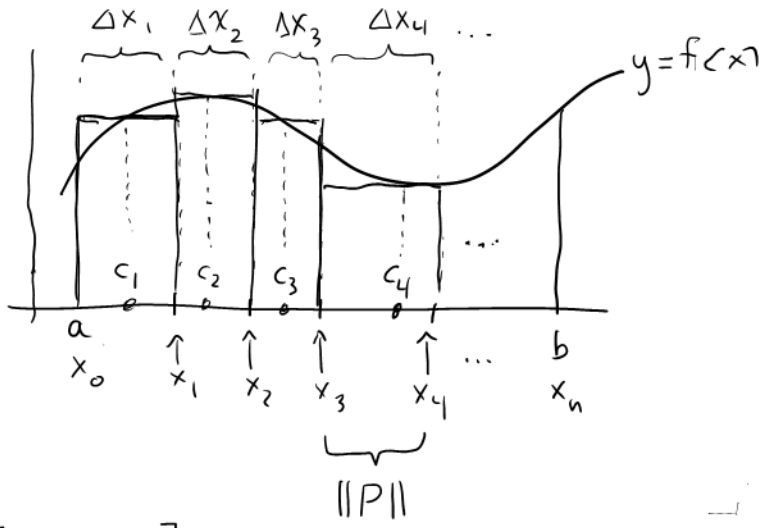
Area of kth rectangle is (height)(base) $= f(x_k)\Delta x$

$$A \approx \sum_{k=1}^{n} f(x_k)\Delta x$$

③ $$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x}$$ ← details of working out limit depend on $f(x)$, a and b.

A more versital approach

Select $x_0\ x_1\ x_2 \cdots x_n$, not necessarily equally spaced. This sequence is called a <u>partition</u> P of the interval $[a,b]$.



Length of $[x_{k-1}, x_k]$ is $x_k - x_{k-1} = \Delta x_k$

<u>Norm</u> of P is $\|P\|$ = largest Δx_k

Put a <u>sample point</u> c_k in each $[x_{k-1}, x_k]$

For each k, make a rectangle with base $[x_{k-1}, x_k]$, height $f(c_k)$

Area of rectangle #k is $f(c_k)\Delta x_k$

$$A \approx \sum_{k=1}^{n} f(c_k)\Delta x_k$$ ← called a <u>Riemann Sum</u>

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$ ← <u>area under curve</u>

☆ **Definition** Given a function $f(x)$ defined on $[a,b]$, the definite integral of $f(x)$ over $[a,b]$ is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

← where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

$$= \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

(provided such limits exist).

Theorem If $f(x)$ is continuous on $[a,b]$ then $\int_a^b f(x)\,dx$ exists.

Don't worry about the proof of this. If you're really curious, take a course in Advanced Calculus (MATH 407). That's the proper place for such issues.

Note

Definite Integral $\int_a^b f(x)\,dx$ is a number

If $f(x) \geq 0$ on $[a,b]$, this # is area under $f(x)$ between a and b

Indefinite Integral $\int f(x)\,dx$ is a function

Or, more accurately it is a set of functions $F(x) + \underline{\underline{C}}$

Example $\int_0^2 x^2\,dx = \frac{8}{3}$ (By what we did earlier) NUMBER

$\int x^2\,dx = \frac{x^3}{3} + C$ FUNCTION

The relationship between the number $\int_a^b f(x)\,dx$ and the function $\int f(x)\,dx$ will be spelled out by the Fundamental Theorem of Calculus, in Section 5.4.