Section 4.6 Applied Optimization

Now we are going to apply the optimization techniques we've learned to real-world problems. But in reality, the kinds of problems calculus is used to solve are much more complex than the ones we consider here. However, we need to start simply. Some of the problems we solve are going to seem somewhat contrived and oversimplified. Keep in mind that the ideas we use here are more important than the problems we solve. In real-life applications of calculus, the problems are much more complex. But the underlying ideas used in their solutions have their origins in exactly the kind of thinking we employ here.

We begin with a ...

Summary of Methods for Finding Absolute Extrema

To find absolute extrema of $f(x)$ on a closed interval $[a,b]$

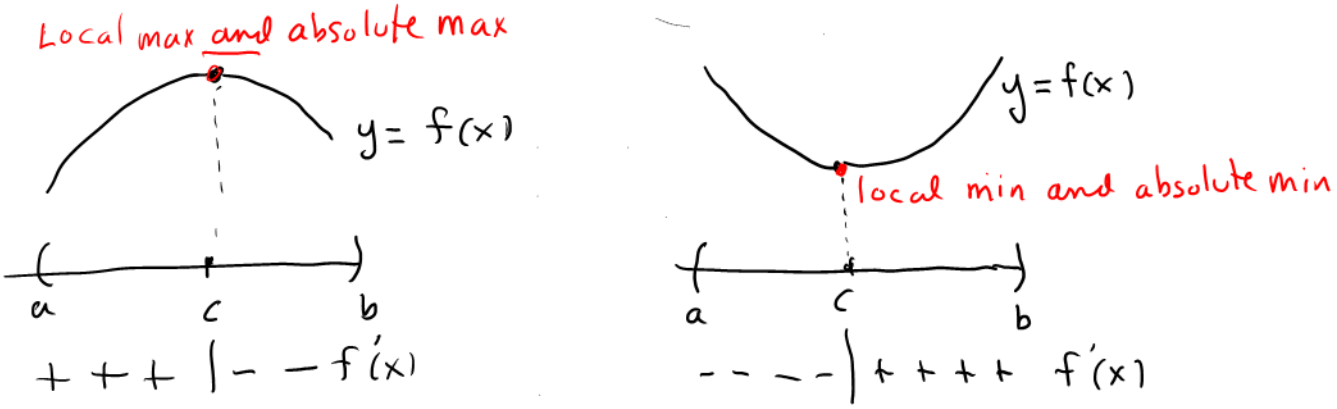
1. Find all critical points c of $f(x)$ in $[a,b]$.
2. Compute $f(c)$ for each critical point c from Step 1. Compute $f(a)$ and $f(b)$.
3. Largest value from step 2 is absolute maximum
   Smalles value is absolute minimum

If the interval on which you seek extrema is not closed, then things are not as clear cut. You need to find the local extrema and analyze increase/decrease of $f(x)$. Maybe draw a rough sketch of the graph of $f(x)$. Then reach a conclusion about absolute extrema based on this information.

However, there is one situation in which the process is quite simple:

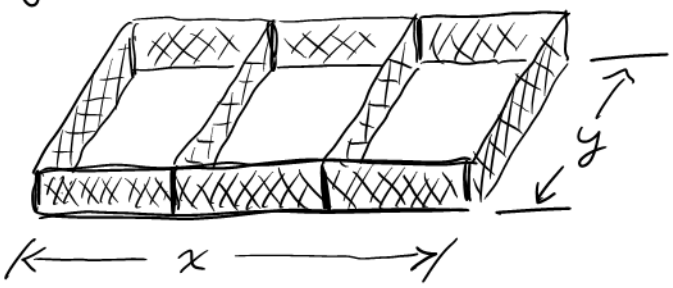
Suppose $f(x)$ has only one critical point c on an interval (open or closed)

- If $f(x)$ has a local max at c, then it's an absolute max
- If $f(x)$ has a local min. at c, then it's an absolute min.



Example

200 feet of chain link fencing is used to enclose three rectangular regions, as illustrated.

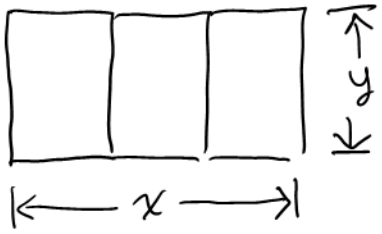


What dimensions x and y give the largest enclosed area?

Note $2x + 4y = 200$

$$4y = 200 - 2x$$

$$y = 50 - \frac{1}{2}x$$



We want to maximize area $= A = xy = x\left(50 - \frac{1}{2}x\right)$

Thus area $= A(x) = x\left(50 - \frac{1}{2}x\right)$

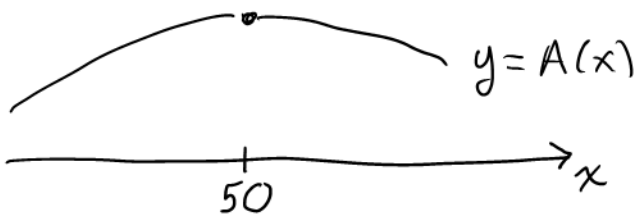
$$A(x) = 50x - \frac{1}{2}x^2$$

Find x that maximizes this

$$A'(x) = 50 - x = 0$$

$$x = 50$$

critical point



$+ + + + + + | - - - -$ $A'(x) = 50 - 2x$

Conclusion.

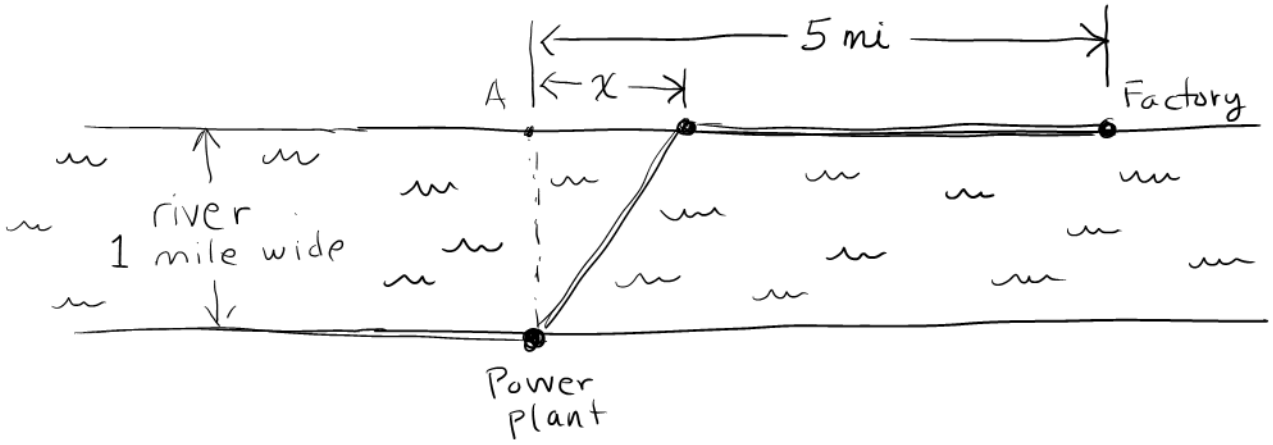
Area maximized when $x = 50$

By (*) above, $y = 50 - \frac{1}{2} \cdot 50 = 25$

Answer Dimensions $x = 50$, $y = 25$ give greatest area

Example

A power plant and a factory are located on opposite banks of a river, as illustrated.



Power line must be laid as illustrated partly underwater, partly above ground along opposite bank.

Cost of underwater cable: \$50 thousand per mile
Cost of surface cable: \$40 thousand per mile

Find x for which construction cost is minimized.

Solution Length of underground cable: $\sqrt{1+x^2}$ miles
Length of surface cable $5-x$ miles

Total cost: $C(x) = 50\sqrt{1+x^2} + 40(5-x)$

$C(x) = 50\sqrt{1+x^2} + 200 - 40x$ ← minimize this on interval $[0,5]$

$$C'(x) = 50\,\frac{1}{2}(1+x^2)^{-1/2}\,2x + 0 - 40$$

$$C'(x) = \frac{50x}{\sqrt{1+x^2}} - 40 = 0$$

$$\frac{50x}{\sqrt{1+x^2}} = 40$$

$$5x = 4\sqrt{1+x^2}$$

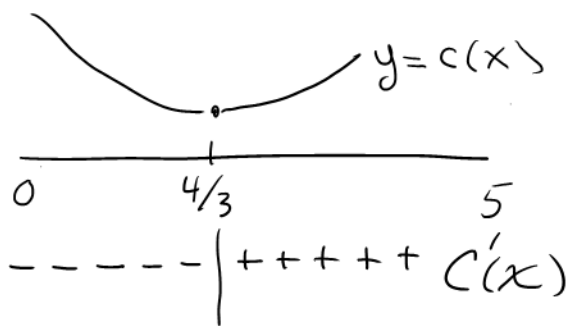
$$25x^2 = 16(1+x^2)$$

$$25x^2 = 16 + 16x^2$$

$$9x^2 = 16$$

$$x^2 = \frac{16}{9}$$

$$x = 4/3$$ ← critical point



Answer

Cost is minimized when $x = 4/3$ mile