Chapter 4 Applications of Derivatives

Now that we know how to find derivatives, and what they mean, we turn to some applications. The main topic of chapter 4 involves finding maximum and minimum values of functions.

Examples Profit $= P(x)$ ← Goal find x that maximizes this.
Cost $= C(x)$ ← Goal find x that minimizes this

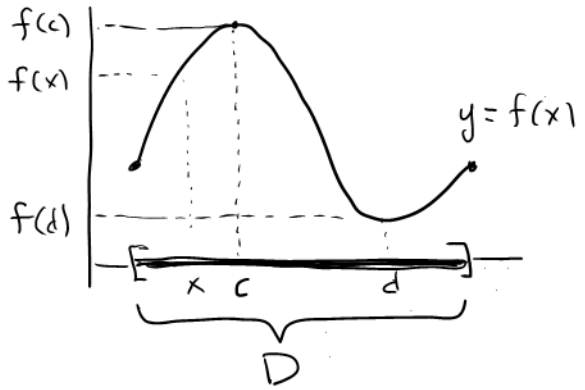
In general, if $f(x)$ is a good or desirable quantity, we want to find an x that maximizes $f(x)$ (i.e. makes it as large as possible). If $f(x)$ is a bad or undisrable quantity, we seek an x that minimizes it (i.e. makes it as small as possible). Thus our goal is to find an optimal outcome. Derivatives (as we will see) provide a means of doing this. Section 4.1 sets up some main ideas and definitions.

Section 4.1 Extreme Values of Functions

Definitions Suppose $f(x)$ is defined on a domain D (usually an interval).

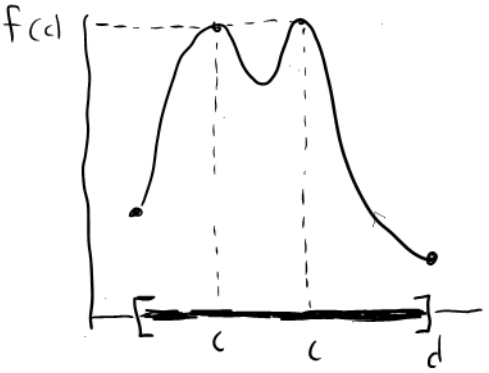
- $f(x)$ has an absolute maximum at c in D if $f(x) \le f(c)$ for all x in D.
- $f(x)$ has an absolute minimum at c in D if $f(x) \ge f(c)$ for all x in D.

Examples



$f(x)$ has absolute maximum at c because $f(x) \le f(c)$ for all x.

$f(x)$ has absolute minimum at d because $f(x) \ge f(d)$ for all x.



This function has an absolute maximum of $f(c)$, which is attained at two separate values of c.

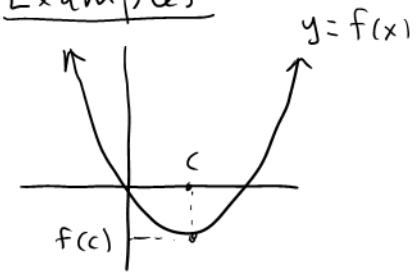
The absolute minimum occurs at d, an endpoint of D.

Terminology:

If $f(x)$ has an abs. max at c, $f(c)$ is an absolute maximum of $f(x)$ at c
If $f(x)$ has an abs. min. at d, $f(d)$ is an absolute minimum of $f(x)$ at d

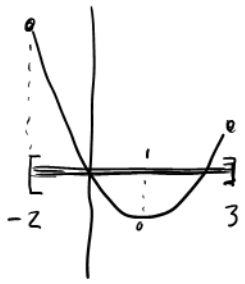
$f(c)$ and $f(d)$ are absolute extrema.

Examples



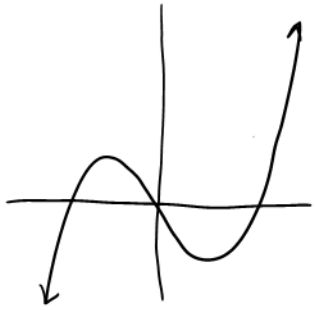
$D = (-\infty, \infty) = \mathbb{R}$

Absolute minimum at c.
No absolute maxmum



$D = [-2, 3]$

Absolute maximum at -2
Absolute minimum at 1



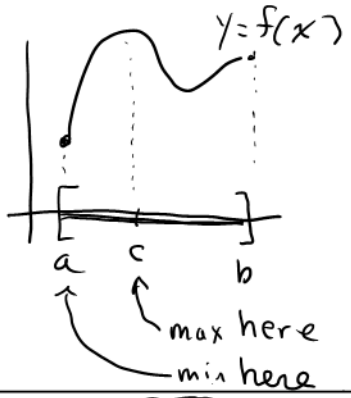
$D = (-\infty, \infty) = \mathbb{R}$

No absolute max.
No absolute min.

So in some circumstances absolute extrema don't exist.
But there is one situation in which it is guaranteed:

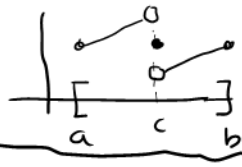
Theorem (Extreme Value Theorem)

If $f(x)$ is continuous on a closed interval $[a, b]$, then it has both an absolute max. and an absolute min. on $[a, b]$



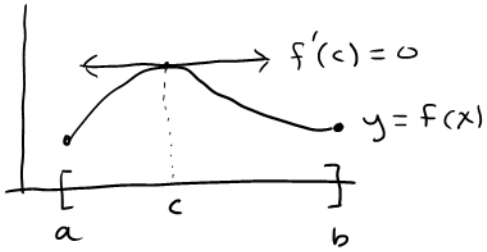
Intuitively this seems plausible, almost obvious, but the proof is subtle and is left to more advaned courses.

Note that the condition of continuity is essential, as this picture suggests:



← no extrema

Our goal is to locate extreme values. Where do they occur? Our pictures suggest that extreme values can happen at endpoints and values of c where $f'(c) = 0$.

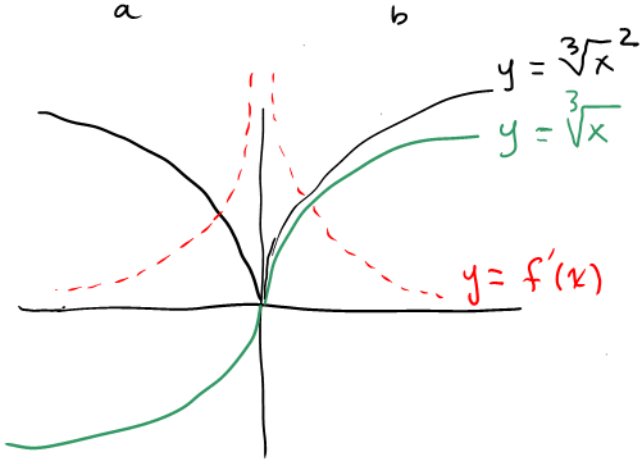


Where else? An example is instructive

Consider $f(x) = \sqrt[3]{x}^{\,2} = x^{2/3}$

$$f'(x) = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$

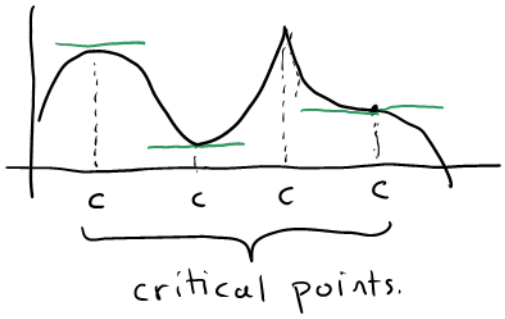
Note that $f(x) = \sqrt[3]{x}^{\,2}$ has an absolute minimum at $c = 0$ and $f'(0)$ is not defined.



Conclusion Extreme values can occur at endpoints or values c for which $f'(c) = 0$ or $f'(c)$ is undefined

This leads to a definition

Definition A number c in the domain of $f(x)$ is called a <u>critical point</u> if $f'(c)=0$ or $f'(c)$ is undefined.



<u>Note</u> critical points are <u>candidates</u> for extreme values.

This leads to our main result of the day:

How to find the extreme values of $f(x)$ on a closed interval $[a,b]$.

1. Find all critical points c in $[a,b]$.
2. Evaluate each $f(c)$ and $f(a)$ and $f(b)$
3. Select the largest value from Step 2. It's the abs. max
   Select the smallest value from Step 2 It's the abs. min.

<u>Example</u> Find absolute extrema of $f(x)=\sqrt[3]{x}^2-\frac{4}{3}x$ on $[0,27]$

1. Find critical pts. of $f(x)$

To do this, examine $f'(x)=\frac{2}{3\sqrt[3]{x}}-\frac{4}{3}$

Note $f'(0)$ is undefined, so 0 is a critical point.

The other critical points will be those $x=c$ for which $f'(c)=0$

To find them, set $f'(x)=0$ and solve for x.

$$f'(x)=0$$
$$\frac{2}{3\sqrt[3]{x}}-\frac{4}{3}=0$$
$$\frac{2}{3}\left(\frac{1}{\sqrt[3]{x}}-2\right)=0$$
$$\frac{1}{\sqrt[3]{x}}-2=0$$

$$\frac{1}{\sqrt[3]{x}}=2$$
$$\sqrt[3]{x}=\frac{1}{2}$$
$$x=\left(\frac{1}{2}\right)^3=\frac{1}{8}$$

Thus the critical points are $c=0$ and $c=\frac{1}{8}$.

Note $c=0$ just happens to also be an endpoint of $[0,27]$

2. $f(0)=\sqrt[3]{0}^2-\frac{4}{3}\cdot 0=0$

$f\left(\frac{1}{8}\right)=\sqrt[3]{\frac{1}{8}}^2-\frac{4}{3}\cdot\frac{1}{8}=\frac{1}{4}-\frac{1}{6}=\frac{3}{12}-\frac{2}{12}=\frac{1}{12}$ } critical pts.

$f(0)=0$

$f(27)=\sqrt[3]{27}^2-\frac{4}{3}\cdot 27=9-36=-27$ } end pts.

3. $f\left(\frac{1}{8}\right)=\frac{1}{12}$ is abs. max.

   $f(27)=-27$ is abs. min.