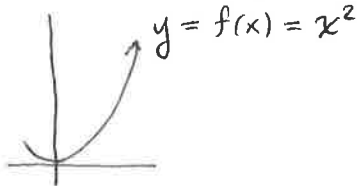
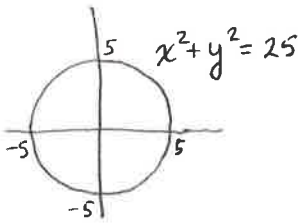
Section 3.7 Implicit Differentiation

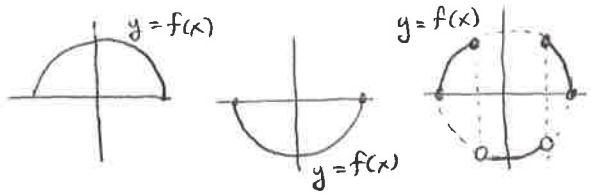
We can now differentiate just about any function $y = f(x)$.



- y is an (explicit) function of x and we can find $f'(x)$.



- But what about this situation?
- Here y is related to x but its not an explicit function of x.
- It's not so clean what the derivative would be

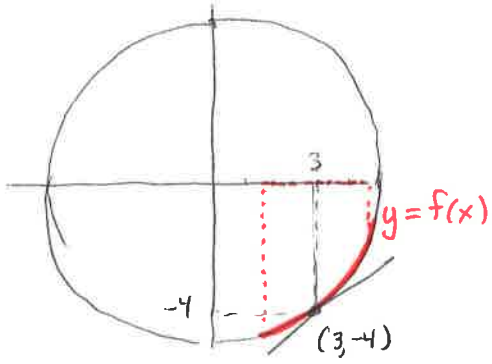


- There are lots of functions $y = f(x)$ satisfying $x^2 + y^2 = 25$.
- They are called <u>implicit</u> functions of the equation $x^2+y^2 = 25$

Todays Goal:
Discover how to differentiate implicit functions

Motivational Problem

Find slope of tangent to graph of $x^2+y^2=25$ at point $(3,-4)$



Let $y = f(x)$ be the above implicit function. The answer will be $f'(3)$

$$x^2 + y^2 = 25$$

$$x^2 + (f(x))^2 = 25$$

$$\frac{d}{dx}\left[x^2 + (f(x))^2\right] = \frac{d}{dx}[25]$$

$$2x + 2f(x)f'(x) = 0$$

$$2f(x)f'(x) = -2x$$

$$f'(x) = \frac{-2x}{2f(x)} = \frac{-x}{f(x)}$$

$$\text{slope} = f'(3) = \frac{-3}{f(3)} = \frac{-3}{-4} = \boxed{\frac{3}{4}}$$

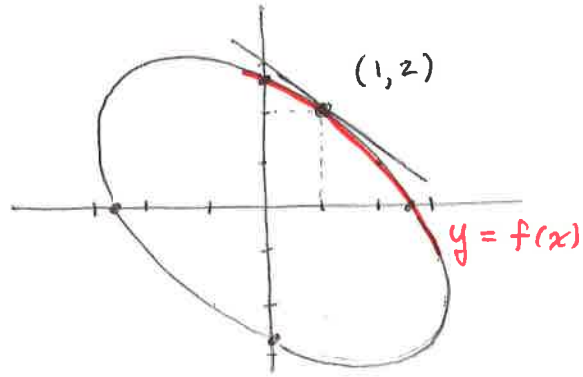
This process is called <u>implicit differentiation</u>

$\sqrt{7} \approx 2.6$

Example

$$x^2 + xy + y^2 = 7$$

Find slope of tangent to the graph at $(1,2)$



Think this: $y = f(x)$

$$x^2 + xy + y^2 = 7$$

$$x^2 + x f(x) + (f(x))^2 = 7$$

$$\frac{d}{dx}\left[x^2 + x f(x) + (f(x))^2\right] = \frac{d}{dx}[7]$$

$$2x + (1)f(x) + x f'(x) + 2 f(x) f'(x) = 0$$

$$x f'(x) + 2 f(x) f'(x) = -2x - f(x)$$

$$f'(x)\big(x + 2f(x)\big) = -2x - f(x)$$

$$f'(x) = \frac{-2x - f(x)}{x + 2f(x)}$$

Slope =

$$f'(1) = \frac{-2\cdot 1 - f(1)}{1 + 2f(1)} = \frac{-2-2}{1+2\cdot 2} = \boxed{-\frac{4}{5}}$$

Do this

$$x^2 + xy + y^2 = 7$$

$$\frac{d}{dx}\left[x^2 + xy + y^2\right] = \frac{d}{dx}[7]$$

$$2x + (1)y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

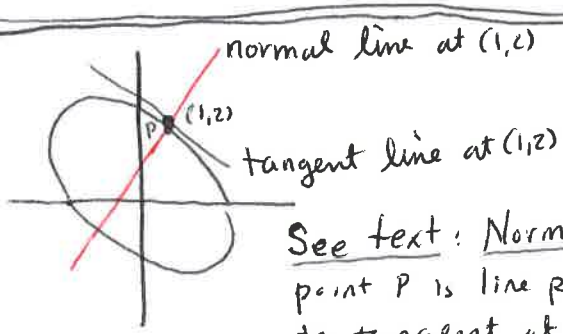
$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x + 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

slope =

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = \frac{-2\cdot 1 - 2}{1 + 2\cdot 2} = \boxed{-\frac{4}{5}}$$



See text: Normal line at a point P is line perpindicular to tangent at P.

Slope of normal line

$$= -\frac{1}{\text{slope of tangent}} = -\frac{1}{-\frac{4}{5}} = \boxed{\frac{5}{4}}$$

Ex

Find the slope of tangent to graph of $\sin xy = \cos xy$ at the point $(\pi, \frac{1}{4})$.

Note: $\sin(\pi \cdot \frac{1}{4}) = \frac{1}{\sqrt{2}} = \cos(\pi \cdot \frac{1}{4})$ so $(\pi, \frac{1}{4})$ is on graph

Solution: $\sin(xy) = \cos(xy)$

$$D_x\left[\sin(xy)\right] = D_x\left[\cos(xy)\right]$$

$$\cos(xy)(y + xy') = -\sin(xy)(y + xy')$$

$$xy'\cos(xy) + y\cos(xy) = -xy'\sin(xy) - y\sin(xy)$$

$$xy'\cos(xy) + xy'\sin(xy) = -y\sin(xy) - y\cos(xy)$$

$$y'(x\cos(xy) + x\sin(xy)) = -y\sin(xy) - y\cos(xy)$$

$$y' = \frac{-y\sin(xy) - y\cos(xy)}{x\cos(xy) + x\sin(xy)} = \frac{-y}{x}$$

Slope at $(\pi, \frac{1}{4})$ is $\frac{-\frac{1}{4}}{\pi} = -\frac{1}{4\pi} \approx -0.08$

$y - \frac{1}{4} = -\frac{1}{4\pi}(x - \pi)$ $\qquad y = -\frac{1}{4\pi}x + \frac{1}{2}$

Graph:

$\sin xy = \cos xy$

$\frac{\sin xy}{\cos xy} = 1$

$\tan(xy) = 1$

$xy = \frac{\pi}{4} + k\pi$

$y = \left(\frac{\pi}{4} + k\pi\right)\frac{1}{x}$