13.4 Solutions for Chapter 13

1. If $x, y \in \mathbb{R}$, then $|x+y| = |x| + |y|$.
This is **false.**
Disproof: Here is a counterexample: Let $x = 1$ and $y = -1$. Then $|x+y| = 0$ and $|x| + |y| = 2$, so it's not true that $|x+y| = |x| + |y|$.

3. If $n \in \mathbb{Z}$ and $n^5 - n$ is even, then n is even.
This is **false.**
Disproof: Here is a counterexample: Let $n = 3$. Then $n^5 - n = 3^5 - 3 = 240$, but n is not even.

5. If A, B, C and D are sets, then $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
This is **false.**
Disproof: Here is a counterexample: Let $A = \{1,2\}$, $B = \{1,2\}$, $C = \{2,3\}$ and $D = \{2,3\}$. Then $(A \times B) \cup (C \times D) = \{(1,1),(1,2),(2,1),(2,2)\} \cup \{(2,2),(2,3),(3,2),(3,3)\} = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$. Also $(A \cup C) \times (B \cup D) = \{1,2,3\} \times \{1,2,3\} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$, so you can see that $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$.

7. If A, B and C are sets, and $A \times C = B \times C$, then $A = B$.
This is **false.**
Disproof: Here is a counterexample: Let $A = \{1\}$, $B = \{2\}$ and $C = \emptyset$. Then $A \times C = B \times C = \emptyset$, but $A \neq B$.

9. If A and B are sets, then $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A - B)$.
This is **false.**
Disproof: Here is a counterexample: Let $A = \{1,2\}$ and $B = \{1\}$. Then $\mathscr{P}(A) - \mathscr{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} - \{\emptyset, \{1\}\} = \{\{2\}, \{1,2\}\}$. Also $\mathscr{P}(A - B) = \mathscr{P}(\{2\}) = \{\emptyset, \{2\}\}$. In this example we have $\mathscr{P}(A) - \mathscr{P}(B) \not\subseteq \mathscr{P}(A - B)$.

11. If $a, b \in \mathbb{N}$, then $a + b < ab$.
This is **false.**
Disproof: Here is a counterexample: Let $a = 1$ and $b = 1$. Then $a + b = 2$ and $ab = 1$, so it's not true that $a + b < ab$.

13. There exists a set X for which $\mathbb{R} \subseteq X$ and $\emptyset \in X$. This is **true.**

Proof. Simply let $X = \mathbb{R} \cup \{\emptyset\}$. If $x \in \mathbb{R}$, then $x \in \mathbb{R} \cup \{\emptyset\} = X$, so $\mathbb{R} \subseteq X$. Likewise, $\emptyset \in \mathbb{R} \cup \{\emptyset\} = X$ because $\emptyset \in \{\emptyset\}$. ■

15. Every odd integer is the sum of three odd integers. This is **true.**

Proof. Suppose n is odd. Then $n = n + 1 + (-1)$, and therefore n is the sum of three odd integers. ■

17. For all sets A and B, if $A - B = \emptyset$, then $B \neq \emptyset$.
This is **false.**
Disproof: Here is a counterexample: Just let $A = \emptyset$ and $B = \emptyset$. Then $A - B = \emptyset$, but it's not true that $B \neq \emptyset$.

19. For every $r, s \in \mathbb{Q}$ with $r < s$, there is an irrational number u for which $r < u < s$. This is **true.**

Proof. (Direct) Suppose $r, s \in \mathbb{Q}$ with $r < s$. Consider the number $u = r + \sqrt{2}\frac{s-r}{2}$. In what follows we will show that u is irrational and $r < u < s$. Certainly since $s - r$ is positive, it follows that $r < r + \sqrt{2}\frac{s-r}{2} = u$. Also, since $\sqrt{2} < 2$ we have

$$u = r + \sqrt{2}\frac{s-r}{2} < r + 2\frac{s-r}{2} = s,$$

and therefore $u < s$. Thus we can conclude $r < u < s$.

Now we just need to show that u is irrational. Suppose for the sake of contradiction that u is rational. Then $u = \frac{a}{b}$ for some integers a and b. Since r and s are rational, we have $r = \frac{c}{d}$ and $s = \frac{e}{f}$ for some $c, d, e, f \in \mathbb{Z}$. Now we have

$$\begin{aligned} u &= r + \sqrt{2}\frac{s-r}{2} \\ \frac{a}{b} &= \frac{c}{d} + \sqrt{2}\frac{\frac{e}{f} - \frac{c}{d}}{2} \\ \frac{ad-bc}{bd} &= \sqrt{2}\frac{ed-cf}{2df} \\ \frac{(ad-bc)2df}{bd(ed-cf)} &= \sqrt{2} \end{aligned}$$

This expresses $\sqrt{2}$ as a quotient of two integers, so $\sqrt{2}$ is rational, a contradiction. Thus u is irrational.

In summary, we have produced an irrational number u with $r < u < s$, so the proof is complete. ■

21. There exist two prime numbers p and q for which $p - q = 97$.
This statement is **false.**
Disproof: Suppose for the sake of contradiction that this is true. Let p and q be prime numbers for which $p - q = 97$. Now, since their difference is odd, p and q must have opposite parity, so one of p and q is even and the other is odd. But there exists only one even prime number (namely 2), so either $p = 2$ or $q = 2$. If $p = 2$, then $p - q = 97$ implies $q = 2 - 97 = -95$, which is not prime. On the other hand if $q = 2$, then $p - q = 97$ implies $p = 99$, but that's not prime either. Thus one of p or q is not prime, a contradiction.

23. If $x, y \in \mathbb{R}$ and $x^3 < y^3$, then $x < y$. This is **true.**

Proof. (Contrapositive) Suppose $x \geq y$. We need to show $x^3 \geq y^3$.
Case 1. Suppose x and y have opposite signs, that is one of x and y is positive and the other is negative. Then since $x \geq y$, x is positive and y is negative. Then, since the powers are odd, x^3 is positive and y^3 is negative, so $x^3 \geq y^3$.
Case 2. Suppose x and y do not have opposite signs. Then $x^2 + xy + y^2 \geq 0$ and

also $x - y \geq 0$ because $x \geq y$. Thus we have $x^3 - y^3 = (x-y)(x^2 + xy + y^2) \geq 0$. From this we get $x^3 - y^3 \geq 0$, so $x^3 \geq y^3$.
In either case we have $x^3 \geq y^3$. ■

25. For all $a, b, c \in \mathbb{Z}$, if $a \mid bc$, then $a \mid b$ or $a \mid c$.
This is **false.**
Disproof: Let $a = 6$, $b = 3$ and $c = 4$. Note that $a \mid bc$, but $a \nmid b$ and $a \nmid c$.

27. The equation $x^2 = 2^x$ has three real solutions.

Proof. By inspection, the numbers $x = 2$ and $x = 4$ are two solutions of this equation. But there is a third solution. Let m be the real number for which $m2^m = \frac{1}{2}$. Then negative number $x = -2m$ is a solution, as follows.

$$x^2 = (-2m)^2 = 4m^2 = 4\left(\frac{m2^m}{2^m}\right)^2 = 4\left(\frac{\frac{1}{2}}{2^m}\right)^2 = \frac{1}{2^{2m}} = 2^{-2m} = 2^x.$$

Therefore we have three solutions 2, 4 and m. ■

29. If $x, y \in \mathbb{R}$ and $|x + y| = |x - y|$, then $y = 0$.
This is **false.**
Disproof: Let $x = 0$ and $y = 1$. Then $|x + y| = |x - y|$, but $y = 1$.

31. No number appears in Pascal's triangle more than four times.
Disproof: The number 120 appears six times. Check that $\binom{10}{3} = \binom{10}{7} = \binom{16}{2} = \binom{16}{14} = \binom{120}{1} = \binom{120}{119} = 120$.

33. Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial of degree 1 or greater, and for which each coefficient a_i is in $\mathbb{N}$. Then there is an $n \in \mathbb{N}$ for which the integer $f(n)$ is not prime.

Proof. (Outline) Note that, because the coefficients are all positive and the degree is greater than 1, we have $f(1) > 1$. Let $b = f(1) > 1$. Now, the polynomial $f(x) - b$ has a root 1, so $f(x) - b = (x-1)g(x)$ for some polynomial g. Then $f(x) = (x-1)g(x) + b$. Now note that $f(b+1) = bg(b) + b = b(g(b) + 1)$. If we can now show that $g(b) + 1$ is an integer, then we have a nontrivial factoring $f(b+1) = b(g(b)+1)$, and $f(b+1)$ is not prime. To complete the proof, use the fact that $f(x) - b = (x-1)g(x)$ has integer coefficients, and deduce that $g(x)$ must also have integer coefficients. ■