CHAPTER 5

Discrete Probability

An urban legend has it that one Friday a weatherman announced "*There's a 50% chance of rain on Saturday, and a 50% chance of rain on Sunday, so there's a 100% chance of rain this weekend.*" Obviously he was wrong, because under the circumstances there's still a chance of no rain at all over the weekend. But what is the correct answer?

Here is one approach to the answer. Make a set of four length-2 lists:

$$S = \{\,\text{RR},\ \ \text{RN},\ \ \text{NR},\ \ \text{NN}\,\}.$$

This set encodes the four possible outcomes for the weather over the weekend. The first letter of each list is either R or N depending on whether there is Rain or NO-rain on Saturday. The second letter is either R or N depending on whether or not there is rain on Sunday. Thus RN means rain on Saturday and no rain on Sunday; NR means no rain on Saturday but rain on Sunday; RR means rain both days; and NN means no rain over the weekend.

The information suggests that each outcome RR, RN, NR and NN is equally likely to occur: There is a 25% chance of RR, a 25% chance of RN, a 25% chance of NR, and a 25% chance of NN.

We want to determine the chance of rain over the weekend. The event of rain over the weekend corresponds to the subset $\{\text{RR}, \text{RN}, \text{NR}\} \subseteq S$.

$$S = \{\,\text{RR},\ \ \text{RN},\ \ \text{NR},\ \ \text{NN}\,\}$$

Thus rain over the weekend will occur in three out of four equally likely outcomes, so the weatherman should have said there is a $\frac{3}{4} = 75\%$ chance of rain over the weekend.

This chapter is about probability and computing the probabilities of events. The above example sets up the main ideas and definitions that are needed. Given a situation with a finite number of possible outcomes (like whether or not there's rain over the weekend), its *sample space* is the set S of all possible outcomes, and an *event* (like rain over the weekend) is a subset of S. Let's set up these ideas carefully.

5.1 Sample Spaces, Events and Probability

In the study of probability, an **experiment** is an activity that produces one of a number of different outcomes that cannot be determined in advance. The **sample space** of the experiment is the set S of all possible outcomes. An **event** is a subset $E \subseteq S$. We say the **event occurs** if the experiment is preformed and the outcome is an element of E.

One example of an experiment was described on the previous page: Observe whether it rains on each day of a weekend, and record the result as one of RR, RN, NR or NN. The sample space of this experiment is the set $S = \{\text{RR, RN, NR, NN}\}$. The event of rain over the weekend is the subset $E = \{\text{RR, RN, NR}\} \subseteq S$. If we preform the experiment and the outcome is one RR, RN or NR, in E, then we say the event E occurs.

There are numerous other events associated with this experiment. The event of rain on Saturday is the subset $E' = \{\text{RR, RN}\} \subseteq S$. Here are some other events $E \subseteq S = \{\text{RR, RN, NR, NN}\}$ for this experiment.

Event		probability of event
Rain over the weekend:	$E = \{\text{RR, RN, NR}\}$	$p(E) = \frac{\lvert E \rvert}{\lvert S \rvert} = \frac{3}{4} = 75\%$
Rain on Sunday:	$E = \{\text{RR, NR}\}$	$p(E) = \frac{\lvert E \rvert}{\lvert S \rvert} = \frac{2}{4} = 50\%$
No rain over weekend:	$E = \{\text{NN}\}$	$p(E) = \frac{\lvert E \rvert}{\lvert S \rvert} = \frac{1}{4} = 25\%$
Rain on just one day:	$E = \{\text{RN, NR}\}$	$p(E) = \frac{\lvert E \rvert}{\lvert S \rvert} = \frac{1}{2} = 50\%$
Nothing happens:	$E = \emptyset$	$p(E) = \frac{\lvert E \rvert}{\lvert S \rvert} = \frac{0}{4} = 0\%$
Something happens:	$E = \{\text{RR, RN, NR NN}\}$	$p(E) = \frac{\lvert E \rvert}{\lvert S \rvert} = \frac{4}{4} = 100\%$

The **probability** or **chance** of an event is the likelihood of its occurring when the experiment is performed. The probability of an event is a number from 0 to 1 (that is, from a 0% chance of occurring to a 100% chance of occurring). We denote the probability of E as $p(E)$. Thus, the experiment of recording the weather over the weekend when there is a 50% chance of rain on each day, the probability of the event $E = \{\text{RR, RN, NR}\}$ is $p(E) = 75\%$, as calculated on the previous page.

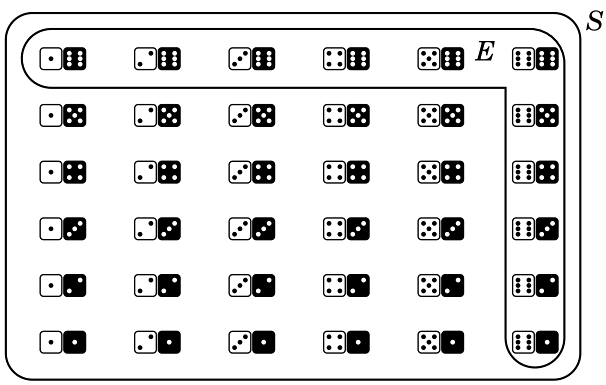
In many cases, all outcomes in a sample space are equally likely to occur. This is the case in the above weekend weather experiment, where each outcome RR, RN, NR, or NN has a 25% chance of occurring. In such a situation, an event E occurs in $|E|$ out of $|S|$ equally likely outcomes, so its probability is $p(E) = \frac{|E|}{|S|}$. See the right-hand column of the above table.

This type of reasoning leads to a formula for the probability of an event when all outcomes in a sample space are equally likely to occur.

> **Fact 5.1** In an experiment where all outcomes in the sample space S are equally likely to occur, the probability of an event $E \subseteq S$ is
>
> $$p(E) = \frac{|E|}{|S|}.$$

Example 5.1 You have two dice, a white one and a black one. You roll both of them. What is the probability that at least one of them will be a six?

Solution. The sample space S is drawn below, showing the 36 equally likely outcomes. The event $E \subseteq S$ of at least one six is also shown.



Note that you will get at least one six in $|E| = 11$ out of $|S| = 36$ equally likely outcomes, so Fact 5.1 says the probability of getting at least one six is

$$p(E) = \frac{|E|}{|S|} = \frac{11}{36} = 0.30\overline{5} = 30.\overline{5}\%.$$

This means that if you roll the pair of dice, say, 100 times, you should expect to get at least one six on about 30 of the rolls. Try it. ✍

Fact 5.1 applies only to situations in which all outcomes in a sample space are equally likely to occur. For an example of an experiment that does not meet this criterion, imagine that one of the dice in Example 5.1 was weighted so that it was more likely to land on six. Then the outcome ⚅⚅ would be more likely than the outcome (say) ⚀⚀, and Fact 5.1 would not apply. In such a case $p(E)$ would be greater than $30.\overline{5}\%$. We will treat this kind of situation in Section 5.4. Until then, all of our experiments will have outcomes that are equally likely, and we will use Fact 5.1 freely.

Example 5.2 You toss a coin three times in a row. What is the probability of getting at least one tail?

Solution. Denote a typical outcome as a length-3 list such as HTH, which means you rolled a head first, then a tail, and then a head. Here is the sample space S and the event E of at least one tail:

$$S = \{\text{HHH}, \overbrace{\text{HHT, HTH, HTT, THH, THT, TTH, TTT}}^{E}\}$$

The chance of getting at least one tail is $p(E) = \frac{|E|}{|S|} = \frac{7}{8} = 0.875 = \mathbf{87.5\%}$. ✍

Example 5.3 You deal a 5-card hand from a shuffled deck of 52 cards. What is the probability that all five cards are of the same suit?

Solution. The sample space S consists of all possible 5-card hands. Such a hand is a 5-element subset of the set of 52 cards, so we could begin writing out S as something like

$$S = \Big\{ \{7\clubsuit, 2\clubsuit, 3\heartsuit, A\spadesuit, 5\diamondsuit\},\ \{8\heartsuit, 2\heartsuit, K\heartsuit, A\heartsuit, 5\heartsuit\}, \ldots\ldots \Big\}.$$

However, this is too big to write out conveniently in its entirety. But note that $|S|$ is the number of ways to select 5 cards from 52 cards, so

$$|S| = \binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52\cdot51\cdot50\cdot49\cdot48}{5\cdot4\cdot3\cdot2\cdot1} = 2,598,960.$$

Now consider the event $E \subseteq S$ consisting of all 5-card hands in S that are of the same suit. We can compute $|E|$ using the addition principle (Fact 4.2 on page 85). The set E can be divided into four parts: the hands that are all hearts, the hands that are all diamonds, the hands that are all clubs and the hands that are all spades.

As the deck has 13 heart cards, the number of 5-card hands that are all hearts is $\binom{13}{5} = \frac{13\cdot12\cdot11\cdot10\cdot9}{5\cdot4\cdot3\cdot2\cdot1} = 1287$. For the same reason, the number of 5-card hands that are all diamonds is also 1287. This is also the number of 5-card hands that are all clubs, and the number of all 5-card hands that are all spades. By the addition principle, $|E| = 1287+1287+1287+1287 = 5148$.

Thus the probability that all cards in the hand are of the same suit is thus $p(E) = \frac{|E|}{|S|} = \frac{5148}{2,598,960} \approx 0.00198 = \mathbf{0.198\%}$.

So in playing cards, you should expect to be dealt a 5-card hand of the same suit only approximately 2 out of 1000 times. ✍

Exercises for Section 5.1

For each problem, write out the sample space S (or describe it if it's too big to write out) and find $|S|$. Then write out or describe the relevant event E. Find $p(E) = \frac{|E|}{|S|}$. You may need to use various counting techniques from Chapter 4

1. A card is randomly selected from a deck of 52 cards. What is the chance that the card is red or a king?

2. A card is randomly selected from a deck of 52 cards. What is the chance that the card is red but not a king?

3. Toss a dice 5 times in a row. What is the probability that you don't get any ⚅'s?

4. Toss a dice 6 times in a row. What is the probability that exactly three of the tosses are even?

5. Toss a dice 5 times in a row. What is the probability that you will get the same number on each roll? (i.e. ⚀⚀⚀⚀⚀ or ⚁⚁⚁⚁⚁, etc.)

6. Toss a dice 5 times in a row. What is the probability that every roll is a different number?

7. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that they show the same number?

8. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that the numbers add up to 7?

9. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that both show even numbers?

10. Toss a coin 8 times. What is the probability of getting exactly two heads?

11. Toss a coin 8 times. Find the probability that the first and last tosses are heads.

12. A hand of four cards is dealt off of a shuffled 52-card deck. What is the probability that all four cards are of the same color? (All red or all black.)

13. Five cards are dealt from a shuffled 52-card deck. What is the probability of getting three red cards and two clubs?

14. A coin is tossed 7 times. What is the probability that there are more tails than heads? What if it is tossed 8 times?

15. Alice and Bob each randomly pick an integer from 0 to 9. What is the probability that they pick the same number? What is the probability that they pick different numbers?

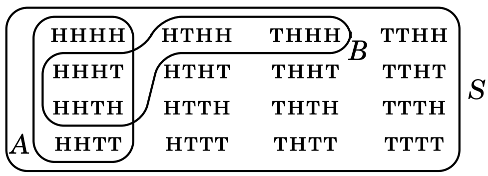
16. Alice and Bob each randomly pick an integer from 0 to 9. What is the probability that Alice picks an even number and Bob picks an odd number?

17. What is the probability that a 5-card hand dealt off a shuffled 52-card deck does not contain an ace?

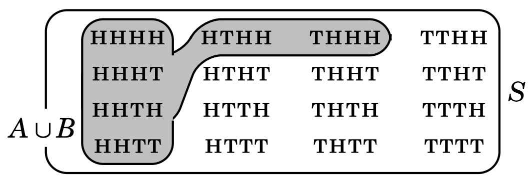
18. What is the probability that a 5-card hand dealt off a shuffled 52-card deck does not contain any red cards?

5.2 Combining Events

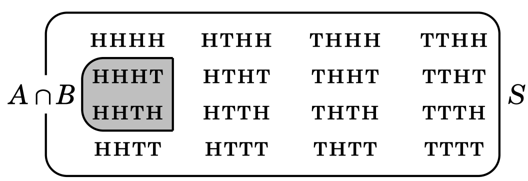
Now we begin combining events. To illustrate this, imagine tossing a coin four times in a row. Let A be the event "*The first two tosses are heads*," and let B be the event "*There are exactly three heads*." The sample space S is shown below, along with the events A and B. Note that $p(A) = \frac{|A|}{|S|} = \frac{4}{16} = 25\%$ and $p(B) = \frac{|B|}{|S|} = \frac{4}{16} = 25\%$.



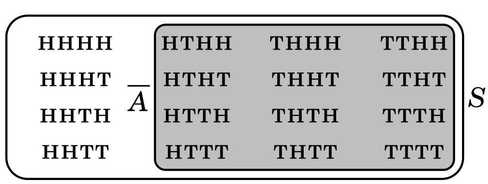
Now, the union $A \cup B$ is a subset of S, so it is an event. Think of it as the event "*The first two tosses are heads* **or** *there are exactly three heads*." This is diagramed below, and we see that $p(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{6}{16} = 37.5\%$.



Also, the intersection $A \cap B$ is a subset of S, so it is an event. It is the event "*The first two tosses are heads* **and** *there are exactly three heads*." This is diagramed below. Note that $p(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{2}{16} = 12.5\%$.



Finally, regard S as a universal set and consider the complement $\overline{A} \subseteq S$, drawn below. This is yet another event. It is the event "*It is not the case that the first two tosses are heads*." We have $p\left(\overline{A}\right) = \frac{|\overline{A}|}{|S|} = \frac{12}{16} = 75\%$.

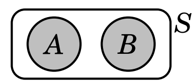


In general, if A and B are events in a sample space, then:

$A \cup B$	is the event	"*A* **or** *B occurs*,"
$A \cap B$	is the event	"*A* **and** *B occur*,"
$\overline{A}$	is the event	"*A* **does not** *occur*."

This section develops formulas for $p(A \cup B)$ and $p\left(\overline{A}\right)$, while the next section treats $p(A \cap B)$. These formulas will be useful because often a complex event has form $E = A \cup B$ or $E = \overline{A}$, where A and B (or $\overline{A}$) are easier to deal with than E. In such cases formulas for $p(A \cup B)$ and $p\left(\overline{A}\right)$ can be handy. But before stating them, we need to lay out a definition.

Definition 5.1 Two events A and B in a sample space S are **mutually exclusive** if $A \cap B = \emptyset$.



Mutually exclusive events have no outcomes in common: If one of them occurs, then the other does not occur. On any trial of the experiment, one of them may occur, or the other, or neither, but *never both*.

Events A and B from the previous page are *not* mutually exclusive, as $A \cap B = \{\text{HHHT}, \text{HHTH}\} \neq \emptyset$. You can toss a coin four times and have both events A: *First two tosses are heads*, and B: *Exactly three heads* occur.

Again, toss a coin four times. Say A is the event "*Exactly three tails*," and B is "*Exactly three heads*." These events are mutually exclusive. You could get three heads, or three tails, or neither (HHTT), but you cannot get three heads **and** three tails in the same four tosses.

Also, if E is any event in a sample space, then E and $\overline{E}$ are mutually exclusive, as $E \cap \overline{E} = \emptyset$. An event E cannot both happen and not happen.

Now we are ready to derive our formula for $p(A \cup B)$. We will get it using Fact 5.1 and the inclusion-exclusion principle (Fact 4.6 on page 104) that states $|A \cup B| = |A| + |B| - |A \cap B|$. Simply observe that

$$\begin{aligned} p(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|} &= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} \\ &= p(A) + p(B) - p(A \cap B). \end{aligned}$$

Note that if A and B happen to be mutually exclusive, then $|A \cap B| = |\emptyset| = 0$, and we get simply $p(A \cap B) = p(A) + p(B)$.

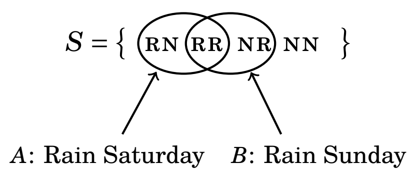
For the formula for $p\left(\overline{A}\right)$, use $\overline{A} = S - A$ and note that

$$p\left(\overline{A}\right) = \frac{|\overline{A}|}{|S|} = \frac{|S - A|}{|S|} = \frac{|S| - |A|}{|S|} = \frac{|S|}{|S|} - \frac{|A|}{|S|} = 1 - p(A).$$

Rearranging $p(\overline{A}) = 1 - p(A)$ gives $p(A) = 1 - p(\overline{A})$, also a useful formula. In summary, we have deduced the following facts.

> **Fact 5.2** Suppose A and B are events in a sample space S. Then:
>
> 1. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
> 2. $p(A \cup B) = p(A) + p(B)$ .... if A and B are mutually exclusive
> 3. $p(\overline{A}) = 1 - p(A)$
> 4. $p(A) = 1 - p(\overline{A})$

Recall our weatherman that we began the chapter with, the one who said that because there was a 50% chance of rain on Saturday and a 50% chance of rain on Sunday, then there was a 100% chance of rain over the weekend. He had only a hazy understanding of the events A: *Rain on Saturday*, and B: *Rain on Sunday*, and their union $A \cup B$: *Rain over the weekend*.



From the data $p(A) = 50\%$ and $p(B) = 50\%$ he concluded $p(A \cup B) = p(A) + p(B) = 50\% + 50\% = 100\%$. The problem is that A and B are not mutually exclusive, as $A \cap B = \{\text{RR}\} \neq \emptyset$. In essence he was using Formula 2 of Fact 5.2, above, when he should have used Formula 1. The correct chance of rain over the weekend, as given by Formula 1, is

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|\{\text{RR}\}|}{|S|} = \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4} = 75\%.$$

Of course we can also get this answer without the aid of the formula $p(A \cup B) = p(A) + p(B) - p(A \cap B)$. Just let $E = A \cup B$ be the event of rain over the weekend. Fact 5.1, which states $p(E) = \frac{|E|}{|S|}$, says $p(E) = \frac{3}{4} = 75\%$. But a word of caution is in order. Recall that Fact 5.1 is only valid in situations in which all outcomes in S are equally likely to occur. Such is the case in this rain-over-the-weekend example (and all other examples in the next three sections), so we do not get into trouble. But the point is that the formulas from Fact 5.2 turn out to hold even if not all outcomes in S are equally likely (even though we derived them under that assumption on the previous page). We will investigate this thoroughly in Section 5.4.

For now, let's do some examples involving Fact 5.2.

Example 5.4 Two cards are dealt from a shuffled deck of 52 cards. What is the probability that both cards are red or both cards are clubs.

Solution. Regard a 2-card hand as a 2-element subset of the set of 52 cards. So the sample space is the set S of 2-element subsets of the 52 cards.

$$S = \left\{ \left\{ \boxed{7\clubsuit}, \boxed{2\clubsuit} \right\}, \left\{ \boxed{8\heartsuit}, \boxed{2\heartsuit} \right\}, \left\{ \boxed{K\spadesuit}, \boxed{2\heartsuit} \right\}, \left\{ \boxed{K\clubsuit}, \boxed{2\heartsuit} \right\}, \ldots \right\}.$$

Though this is too large to write out, we can compute $|S| = \binom{52}{2} = \frac{52\cdot 51}{2} = 1326$.

We are asked to compute $p(E)$ where E is the event

E: *Both cards are red* **or** *both cards are clubs.*

We can decompose E as $E = A \cup B$ where A and B are the events

A: *Both cards are red* $\qquad$ B: *Both cards are clubs*

Thus $A = \left\{ \left\{ \boxed{K\heartsuit}, \boxed{K\diamondsuit} \right\}, \left\{ \boxed{K\heartsuit}, \boxed{2\heartsuit} \right\}, \left\{ \boxed{8\diamondsuit}, \boxed{2\heartsuit} \right\}, \left\{ \boxed{Q\diamondsuit}, \boxed{K\diamondsuit} \right\}, \ldots \right\} \subseteq S,$

and $B = \left\{ \left\{ \boxed{K\clubsuit}, \boxed{5\clubsuit} \right\}, \left\{ \boxed{7\clubsuit}, \boxed{8\spadesuit} \right\}, \left\{ \boxed{J\clubsuit}, \boxed{Q\clubsuit} \right\}, \left\{ \boxed{J\clubsuit}, \boxed{7\clubsuit} \right\}, \ldots \right\} \subseteq S.$

Note that these two events are mutually exclusive, as club cards are black. Further, $|A| = \binom{26}{2} = 325$ because to make a 2-card hand of red cards we have to choose 2 of the 26 red cards. Also, $|B| = \binom{13}{2} = 78$ because to make a 2-card hand of club cards we have to choose 2 of the 13 clubs. Using Formula 2 from Fact 5.2 as well as Fact 5.1 (page 132), our answer is

$$\begin{aligned} p(E) = p(A \cup B) = p(A) + p(B) &= \frac{|A|}{|S|} + \frac{|B|}{|S|} \\ &= \frac{325}{1326} + \frac{78}{1326} = \frac{403}{1326} \approx 0.3039 = \mathbf{30.39\%}. \end{aligned}$$ ✍

You may prefer to solve this example without using Fact 5.2. Instead you can use Fact 5.1 combined with the addition principle, $|A \cup B| = |A| + |B|$, which holds when $A \cap B = \emptyset$ (as is the case here because A and B are mutually exclusive). Then compute the answer as

$$p(E) = p(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B|}{|S|} = \frac{325 + 78}{1326} = \frac{403}{1326} \approx \mathbf{30.39\%}.$$

But as noted on the previous page, there will be situations where Fact 5.2 is unavoidable. So it is not advisable to always bypass it. For now your best strategy is to become accustomed to it, but at the same time be on the lookout for alternate methods.

Example 5.5 Two cards are dealt from a shuffled deck of 52 cards. What is the probability both cards are red or both cards are face cards (J,K,Q)?

Solution. As before, the sample space is the set S of 2-element subsets of the 52 cards, and $|S| = \binom{52}{2} = 1326$:

$$S = \left\{ \left\{ \boxed{7\clubsuit}, \boxed{2\clubsuit} \right\},\ \left\{ \boxed{8\heartsuit}, \boxed{2\heartsuit} \right\},\ \left\{ \boxed{K\spadesuit}, \boxed{2\heartsuit} \right\},\ \left\{ \boxed{K\clubsuit}, \boxed{2\heartsuit} \right\}, \ldots \right\}.$$

We are asked to compute $p(E)$ where E is the event

E: *Both cards are red or both cards are clubs.*

We can decompose E as $E = A \cup B$ where A and B are the events

A: *Both cards are red*
B: *Both cards are face cards*

Let's take a moment to diagram these two events, and their intersection.

$$\begin{aligned}
A &= \left\{ \left\{ \boxed{K\heartsuit}, \boxed{K\diamondsuit} \right\},\ \left\{ \boxed{K\heartsuit}, \boxed{2\heartsuit} \right\},\ \left\{ \boxed{8\diamondsuit}, \boxed{2\heartsuit} \right\},\ \left\{ \boxed{Q\diamondsuit}, \boxed{K\diamondsuit} \right\}, \ldots \right\} \subseteq S. \\
B &= \left\{ \left\{ \boxed{K\heartsuit}, \boxed{K\diamondsuit} \right\},\ \left\{ \boxed{K\spadesuit}, \boxed{J\spadesuit} \right\},\ \left\{ \boxed{J\diamondsuit}, \boxed{Q\heartsuit} \right\},\ \left\{ \boxed{Q\diamondsuit}, \boxed{J\clubsuit} \right\}, \ldots \right\} \subseteq S. \\
A \cap B &= \left\{ \left\{ \boxed{K\heartsuit}, \boxed{K\diamondsuit} \right\},\ \left\{ \boxed{K\diamondsuit}, \boxed{Q\diamondsuit} \right\},\ \left\{ \boxed{J\diamondsuit}, \boxed{Q\heartsuit} \right\},\ \left\{ \boxed{Q\diamondsuit}, \boxed{J\diamondsuit} \right\}, \ldots \right\} \subseteq S.
\end{aligned}$$

Note that A and B are not mutually exclusive, because $A \cap B \neq \emptyset$. (It is possible for the two cards to be *both* red *and* both face cards.) Also,

$$|A| = \binom{26}{2} = \frac{26 \cdot 25}{2} = 325 \qquad \text{(choose 2 out of 26 red cards)}$$

$$|B| = \binom{12}{2} = \frac{12 \cdot 11}{2} = 66 \qquad \text{(choose 2 out of 12 face cards)}$$

$$|A \cap B| = \binom{6}{2} = \frac{6 \cdot 5}{2} = 15 \qquad \text{(choose 2 out of 6 red face cards)}$$

Using Fact 5.2, we get

$$\begin{aligned}
p(E) = p(A \cup B) = p(A) + p(B) - p(A \cup B) &= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} \\
&= \frac{325}{1326} + \frac{66}{1326} - \frac{15}{1326} = \frac{376}{1326} \approx 0.2835 = \mathbf{28.35\%}.
\end{aligned}$$

✍

Example 5.6 Two cards from dealt off a shuffled deck of 52 cards. What is the probability they are not both red?

Solution. The sample space is the set S of 2-element subsets of the 52 cards, and $|S| = \binom{52}{2} = 1326$:

$$S = \left\{ \underbrace{\left\{\boxed{7\clubsuit}, \boxed{2\clubsuit}\right\}, \left\{\boxed{K\spadesuit}, \boxed{2\heartsuit}\right\}, \left\{\boxed{8\heartsuit}, \boxed{2\spadesuit}\right\} \cdots}_{E:\ \textit{Not both red}} \underbrace{\left\{\boxed{K\heartsuit}, \boxed{2\heartsuit}\right\}, \left\{\boxed{K\diamondsuit}, \boxed{2\heartsuit}\right\} \cdots}_{\overline{E}:\ \textit{Both red}} \right\}.$$

We need to compute the probability of the event E: *Not both cards are red*. This event contains pairs of cards that are both black, as well as those for which one card is red and the other is black. The event $\overline{E}$ is simpler. It is the set of all elements of S for which it is not the case that not both cards are red. In other words, $\overline{E}$ is the event $\overline{E}$: *Both cards are red*.

It is easy to compute the cardinality of $\overline{E}$. It is $|\overline{E}| = \binom{26}{2} = 325$, the number of ways to choose 2 cards from the 26 red cards. Fact 5.2 now gives our solution:

$$p(E) = 1 - p\left(\overline{E}\right) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{325}{1326} = \frac{1326 - 325}{1326} = \frac{1001}{1326} \approx 0.7549 = \mathbf{75.49}\%.$$

That is the answer, but before moving on, let's redo the problem using a different approach. The event E is the union of the mutually exclusive events A: *Both cards are black*, and B: *One card is black and the other is red.* Here $|A| = \binom{26}{2} = 325$, the number of ways to choose 2 cards from the 26 blacks, while the multiplication principle says $|B| = 26 \cdot 26 = 676$ (chose a black card and then choose a red one). Fact 5.2 gives

$$p(E) = p(A \cup B) = p(A) + p(B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} = \frac{325}{1326} + \frac{676}{1325} = \frac{1001}{1326} = \mathbf{75.49}\%.$$

Shuffle a 52-card deck; deal two cards, put them back. Repeat 100 times. On about 75 of the trials, not both cards will be red. ✍

But what if you shuffled the deck, dealt two cards, *but did not put them back*. Then you deal two cards from the remaining 50 cards. Is there still a 75.49% chance of not getting two reds? Does the outcome of the first trial affect the probability of the second? The next section investigates this kind of question.

Exercises for Section 5.2

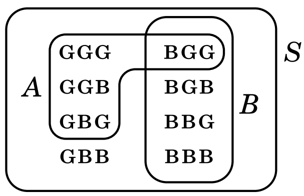
1. A card is taken off the top of a shuffled 52-card deck. What is the probability that it is black or an ace?
2. What is the probability that a 5-card hand dealt off a shuffled 52-card deck contains at least one ace?
3. What is the probability that a 5-card hand dealt off a shuffled 52-card deck contains at least one red card?
4. A hand of five cards is dealt off a shuffled 52-card deck. What is the probability that the five cards are not all of the same suit?
5. You toss a fair coin 8 times. What is the probability that you do not get 4 heads?
6. A 4-card hand is dealt off a shuffled 52-card deck. What is the probability that the cards are all of the same color (i.e. all red or all black)?
7. Two cards are dealt off a shuffled 52-card deck.What is the probability that the cards are both red or both aces?
8. A coin is tossed six times. What is the probability that the first two tosses are heads or the last toss is a head?
9. A dice is tossed six times. You win \$1 if the first toss is a five or the last toss is even. What are your chances of winning?
10. A box contains 3 red balls, 3 blue balls, and 3 green ball. You reach in and grab 2 balls. What is the probability that they have the same color?
11. A dice is rolled 5 times. Find the probability that not all of the tosses are even.
12. Two cards are dealt off a well-shuffled deck. You win \$1 if either both cards are red or both cards are black. Find the probability of your winning.
13. Two cards are dealt off a well-shuffled deck. You win \$1 if the two cards are of different suits. Find the probability of your winning?
14. A dice is tossed six times. You win \$1 if there is at least one ⚄. Find the probability of winning.
15. A coin is tossed 5 times. What is the probability that the first toss is a head or exactly 2 out of the five tosses are heads?
16. In a shuffled 52-card deck, what is the probability that the top card is black or the bottom card is a heart?
17. In a shuffled 52-card deck, what is the probability that neither the top nor bottom card is a heart?
18. A bag contains 20 red marbles, 20 green marbles and 20 blue marbles. You reach in and grab 15 marbles. What is the probability of getting 5 of each color?
19. A bag contains 20 red marbles, 20 green marbles and 20 blue marbles. You reach in and grab 15 marbles. What is the probability that they are all the same color?

5.3 Conditional Probability and Independent Events

Sometimes the probability of one event A will change if we know that another event B has occurred. This is what is known as *conditional probability*. Here is an illustration.

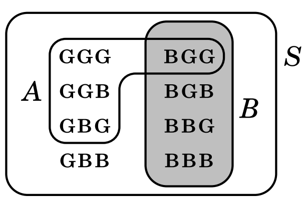
Imagine starting a family of three children. Assume the probability of having a boy is 50% and the probability of having a girl is 50%. What is more likely: the family has more girls than boys, or the oldest is a boy?

To decide, we write out the sample space for the family, listing the possible outcomes of having three children. Let GBB mean a girl was born first, then a boy, and then another boy. Likewise, BGB means a boy was born first, then a girl, then a boy, etc. The 8 equally-likely outcomes are shown below, with events A: *More girls than boys*, and B: *Oldest is a boy*.



Note $p(A) = \frac{|A|}{|S|} = \frac{4}{8} = 50\%$, and $p(B) = \frac{|B|}{|S|} = \frac{4}{8} = 50\%$, so more girls than boys is just as likely as the oldest being a boy.

But now imagine that the event B has occurred, so the oldest is a boy. *Now* what is the probability of more girls than boys? That is, what is $p(A)$? There are just four outcomes in event B, and for only one of them are there more girls than boys. Thus, given this new information (oldest is a boy) $p(A)$ has changed value to $p(A) = \frac{1}{4} = 25\%$.



So we have a situation in which $p(A) = 50\%$, but under the condition that B has occurred, then $p(A) = 25\%$. We express this as $p(A|B) = 25\%$, which we read as "*the conditional probability of A given that B has occurred is* 25%," or just "*the conditional probability of A given B is* 25%."

Definition 5.2 If A and B are two events in a sample space, then the **conditional probability of A given B**, written $P(A|B)$, is the probability that A will occur if B has already occurred.

Example 5.7 Toss a coin once. The sample space is $S = \{\text{H}, \text{T}\}$. Consider the events $A = \{\text{H}\}$ of getting a head and $B = \{\text{T}\}$ of getting a tail. Then $p(A) = p(B) = 50\%$, but $p(A|B) = 0$ because if B (tails) has happened, then A (heads) will not happen. Also $p(B|A) = 0$. Note that $p(A|A) = 1 = 100\%$.

Example 5.8 Draw one card from a shuffled deck, and then draw another. Consider the following events.

A :	*The first card is a heart*	C :	*The second card is a heart*
B :	*The first card is black*	D :	*The second card is a red*

Find $p(A)$, $p(B)$, $p(D)$, $p(C|A)$, $p(A|C)$, $p(A|B)$, $p(D|C)$ and $p(C|D)$.

Solution. All answers can be found without considering the sample space S. For example, $p(A) = \frac{13}{52} = \frac{1}{4}$ because in dealing the first card there are 13 hearts in the 52 equally-likely cards. But to be clear, note that S is the set of all non-repetitive length-2 lists whose entries are cards in the deck. The first list entry is the first card drawn; the second entry is the second card. Drawing 7♣ and then 2♣ is a different outcome than 2♣ followed by 7♣.

$$S = \left\{ \boxed{7\clubsuit}\boxed{2\clubsuit},\ \boxed{2\clubsuit}\boxed{7\clubsuit},\ \boxed{K\spadesuit}\boxed{8\heartsuit},\ \boxed{5\clubsuit}\boxed{2\heartsuit},\ \boxed{5\clubsuit}\boxed{2\heartsuit}, \ldots \right\}$$

Compare this to Example 5.6, where the outcomes were 2-element *sets*, not lists. In the present case $|S| = P(52,2) = 52 \cdot 51 = 2652$.

As noted above, $p(A) = \frac{13}{52} = \frac{1}{4}$ because in dealing the first card there are 13 hearts in the 52 cards. Alternatively, the event $A \subseteq S$ consists of all the 2-elements lists whose first entry is a heart. There are 13 choices for the first entry, and then 51 of the remaining cards can be selected for the second entry. Thus $|A| = 13 \cdot 51$, and $p(A) = \frac{|A|}{|S|} = \frac{13 \cdot 51}{52 \cdot 51} = \frac{1}{4}$.

Similarly, $\boxed{p(B) = \frac{26}{52} = \frac{1}{2}}$ because in drawing the first card, there are 26 black cards out of 52. This could also be done by computing $|B|$, as above.

Evidently, $p(D) = \frac{1}{2}$, as half the cards are red. Alternatively, D consists of all the lists in S whose second entry is red, so by the multiplication principle, $|D| = 51 \cdot 26$. (Fill in the red second entry first, and then put one of the remaining 51 cards in the first entry.) Then $\boxed{p(D) = \frac{|D|}{|S|} = \frac{51 \cdot 26}{52 \cdot 51} = \frac{1}{2}.}$

Note $\boxed{p(C|A) = \frac{12}{51}}$ because if A has occurred, then a heart was drawn first, and there are 12 remaining hearts out of 51 cards for the second draw.

For $p(A|C)$, imagine the two cards have been dealt, one after the other, face down. The second card is turned over, and it is a heart. Event C has occurred. Now what is the chance that A occurred? That is, what is the chance that first card—when turned over—is a heart? It is not the second card, and there are 51 other cards, and 12 of them are hearts. Thus the chance that the first card is a heart is $\frac{12}{51}$, so $\boxed{p(A|C) = \frac{12}{51}.}$

Also, $\boxed{p(A|B) = 0}$ because if B occurs (first card black), then A (first card heart) is impossible. Finally, $\boxed{p(D|C) = 100\%,}$ but $\boxed{p(C|D) = 50\%}$ because if the second card is red, there is a one in two chance that it is a heart. ✍

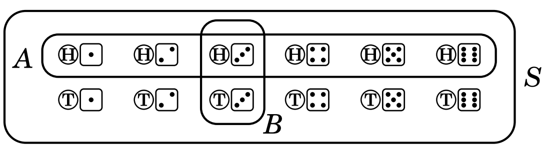
We will soon derive formulas for conditional probability, but they involve a definition that is motivated by the next example of two events having no bearing on one another.

Example 5.9 Toss a coin and roll a dice. Consider the following events.

A : *Coin is heads* B : *Dice is* ⚂

Find $p(A)$, $p(B)$, $p(A|B)$ and $p(B|A)$.

Solution. Common sense says $p(A) = \frac{1}{2}$ and $p(B) = \frac{1}{6}$. Also, getting ⚂ has no bearing the coin's outcome, and vice versa, so $P(A|B) = \frac{1}{2}$ and $p(B|A) = \frac{1}{6}$. Nonetheless, let's work it out carefully. The sample space S and events A and B are diagramed below.



We see that $P(A) = \frac{|A|}{|S|} = \frac{6}{12} = \frac{1}{2}$ and $P(B) = \frac{|B|}{|S|} = \frac{2}{12} = \frac{1}{6}$.

To find $p(A|B)$, imagine that B has occurred. Now what is the chance that A occurs? Only one of the two outcomes in B is heads, so $p(A|B) = \frac{1}{2}$.

To find $p(A|B)$, imagine that A has occurred. Now what is the chance that B occurs? Only 1 of the 6 outcomes in A has ⚂, so $p(B|A) = \frac{1}{6}$. ✍

In the above example, whether or not B happens has no bearing on the probability of A, and vice versa. We say that events A and B are *independent*.

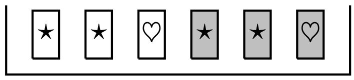
Definition 5.3 Two events A and B are **independent** if one happening does not change the probability of the other happening, that is, if $p(A) = p(A|B)$ and $p(B) = p(B|A)$. Otherwise they are **dependent**.

Thus events A and B in Example 5.9 are independent.

In Example 5.8 we dealt two cards off a deck. For events A: *First card* ♡, and B: *First card black*, we saw $p(A) = \frac{1}{4}$ and $p(A|B) = 0$, so A and B are dependent. (In fact, they also happen to be mutually exclusive.)

We began this section showing that in a family of three children, the event A : *More girls than boys* has $p(A) = 50\%$. But if B: *Oldest is a boy* occurs, then $p(A|B) = 25\%$. Here A and B are dependent. (But note that they are not mutually exclusive).

Example 5.10 A box contains six tickets, three white and three gray, and marked as shown below. You reach in and grab a ticket at random.



Consider events A: *Ticket is gray*, and B: *Ticket has a star on it*. Are these events independent or dependent?

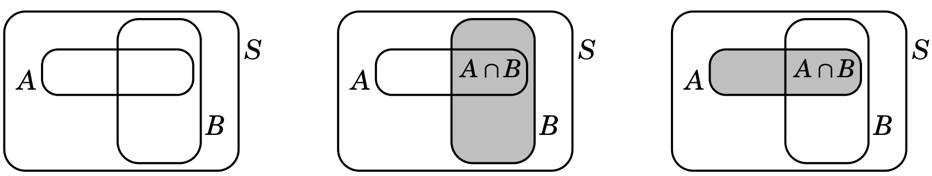
Solution. The chance of getting a gray ticket is $p(A) = \frac{3}{6} = \frac{1}{2}$. The chance of getting a star is $p(B) = \frac{4}{6} = \frac{2}{3}$.

If B occurs, then one of the four tickets with a star has been drawn. Half of these are gray, so $p(A|B) = \frac{1}{2}$, and this equals $p(A)$.

If A occurs, then one of the three gray tickets has been drawn. Two of these have stars, so $p(B|A) = \frac{2}{3}$, and this equals $p(B)$.

Thus A and B are independent. One of them happening does not change the probability of the other happening. ✍

Now we are going to derive general formulas for $p(A|B)$ and $p(B|A)$. Let A and B be two events in a sample space S, as shown below on the left.



If B occurs (shown shaded in the middle drawing) then any outcome in the shaded region could occur, so the shaded set B is like a new sample space. Now if also A occurs, this means some outcome in $A \cap B$ occurs. Note that $A \cap B \subseteq B$ is an event in B, so Fact 5.1 gives

$$p(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|}{|B|} \cdot \frac{\frac{1}{|S|}}{\frac{1}{|S|}} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{p(A \cap B)}{p(B)}.$$

Thus $p(A|B) = \frac{p(A \cap B)}{p(B)}$. Reversing the roles of A and B (and referring to the drawing on the above right) we also get $p(B|A) = \frac{p(A \cap B)}{p(A)}$. Cross-multiplying gives $p(A \cap B) = p(A|B) \cdot p(B)$ and $p(A \cap B) = p(A) \cdot p(B|A)$.

Thus we have formulas for not only $p(A|B)$ and $p(B|A)$, but also one for $p(A \cap B)$. Moreover, if A and B happen to be independent, then $p(A|B) = p(A)$, so the equation $p(A \cap B) = p(A|B) \cdot p(B)$ simplifies to $p(A \cap B) = p(A) \cdot p(B)$.

Fact 5.3 Suppose A and B are events in a sample space. Then:

1. $p(A|B) = \dfrac{p(A \cap B)}{p(B)}$
2. $p(B|A) = \dfrac{p(A \cap B)}{p(A)}$
3. $p(A \cap B) = p(A|B) \cdot p(B) = p(A) \cdot p(B|A)$
4. $p(A \cap B) = p(A) \cdot p(B)$ ....if A and B are independent.

In the earlier examples in this section, we found conditional probabilities $p(A|B)$ and $p(B|A)$ without the aid of the above formulas. In fact, it turns out that the above formulas 1 and 2 are of relatively limited use. But their consequences, formulas 3 and 4 are very useful, as they provide a method of computing $p(A \cap B)$, the probability that A and B both occur.

Example 5.11 Two cards are dealt off a deck. You win \$1 if the first card is red and the second card is black. What are your chances of winning?

Solution. Let A be the event "*The first card is red,*" and let B be the event "*The second card is black.*" We seek $p(A \text{ and } B)$, which is $p(A \cap B)$. Formula 3 above gives $p(A \cap B) = p(A) \cdot p(B|A) = \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{51} \approx 0.2549 = 25.49\%$. ✍

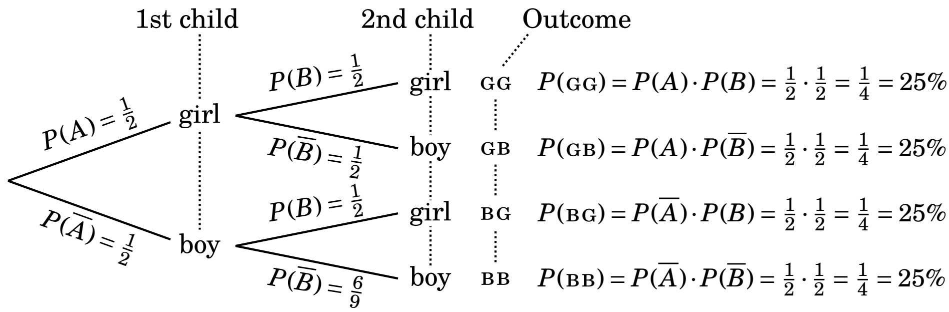
Example 5.12 A dice is rolled twice. You win \$1 if neither roll is ⚀. What are your chances are winning?

Solution. Let A be the event "*The first roll is not* ⚀," so $p(A) = \frac{5}{6}$. Let B be the event "*The second roll is not* ⚀," so $p(B) = \frac{5}{6}$. We seek $p(A \text{ and } B)$.

Events A and B are independent, because the result of the first roll does not influence the second. Formula 4 above gives $p(A \text{ and } B) = p(A \cap B) = p(A) \cdot p(B) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36} = 69.\overline{4}\%$. ✍

Questions about conditional probability can sometimes be answered by a so-called **probability tree**. To illustrate this, suppose (as we assume in this chapter) that there is a 50-50 chance of a child being born a boy or a girl. Suppose a woman has two children. The events A: *First child is a girl,* and B: *Second child is a girl* are independent; whether or not the first child is a girl does not change the probability that the second child is a girl. Thus the chance that both children are girls is $p(A \cap B) = p(A) \cdot p(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 25\%$.

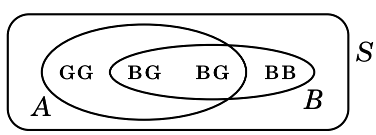
Notice that in this example the complements of A and B are the events $\overline{A}$: *First child is a boy,* and $\overline{B}$: *Second child is a boy.* The probability of the outcome GB (first child is a girl and the second is a boy) is thus $p(\text{GG}) = p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 25\%$. Similarly, we can find the probabilities of all four outcomes in the branches of the following tree.



This confirms our intuitive supposition that each outcome in the sample space $S = \{\text{GG}, \text{GB}, \text{BG}, \text{BB}\}$ has a 25% chance of occurring.

Example 5.13 You meet a woman and a girl. The woman tells you that she has two children, one of whom is the girl. What are the chances that her other child is a boy?

Solution. Most of us would jump to the conclusion that the answer is 50%. But this is wrong. To see why, consider the sample space S, below, for the "experiment" of having two children.



Let A be the event of there being at least one girl. We met a daughter, so A has occurred. Let B be the event of there being a boy in the family. The problem thus asks for the probability of B given that A has occurred. Looking at the above diagram, we see that B occurs in 2 out of the 3 equally likely outcomes in A, so the answer to the question is $p(B|A) = \frac{2}{3} = 66.\overline{6}\%$.

Alternatively, we can use Formula 2 from Fact 5.3 to get the answer as

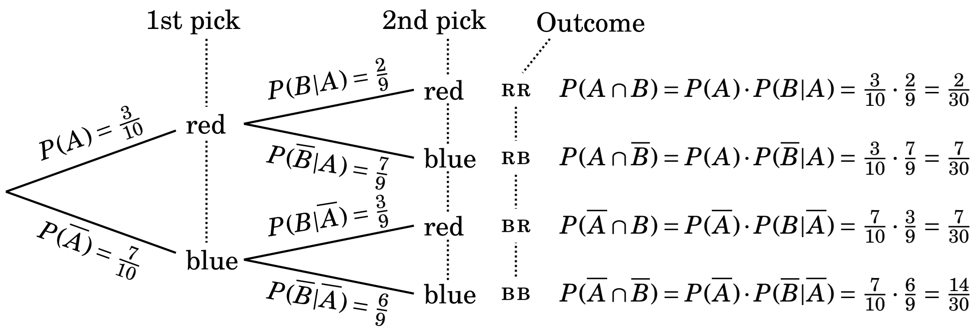
$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3} = 66.\overline{6}\%.$$ ✍

Our next example involves an experiment with a sample space in which not all outcomes are equally likely.

Example 5.14 A jar contains 3 red balls and 7 blue balls. You reach in, pick a ball at random, and remove it. Then you randomly remove a second ball. Thus the sample space for this experiment $S = \{\text{RR, RB, BR, BB}\}$. Find the probability of each outcome in S.

Solution. Form the events A: *First pick is red,* and B: *Second pick is red.* Then we have $\overline{A}$: *First pick is blue,* and $\overline{B}$: *Second pick is blue.*

The probability of the first pick is red is $p(A) = \frac{3}{10}$, as there are 3 out of 10 red balls that we could have picked. Once this has happened, there are 9 balls left, two of which are red, so $p(B|A) = \frac{2}{9}$. So the probability that both picks are red is $p(\text{RR}) = p(A \cap B) = p(A) \cdot p(B|A) = \frac{3}{10} \cdot \frac{2}{9} = \frac{2}{30}$. This is tallied in the top branch of the following tree.



Likewise, $p(\text{RB}) = p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}|A)$. To find $p(\overline{B}|A)$, note that if A has occurred, then there are 9 balls left in the jar, and 7 of them are blue, so $p(A \cap \overline{B}) = \frac{7}{9}$. Thus $p(\text{RB}) = p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}|A) = \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}$, and this is shown in the second-from-the-top branch of the tree.

Similar computations for the probabilities of the remaining two outcomes are shown on the bottom branches. Check that you understand them. From this tree we see that the probabilities of the various outcomes in S are

$$S = \left\{ \begin{array}{cccc} \text{RR}, & \text{RB}, & \text{BR}, & \text{BB} \\ 6.\overline{6}\% & 23.\overline{3}\% & 23.\overline{3}\% & 46.\overline{6}\% \end{array} \right\}$$

✍

If in the above Example 5.14, we had been asked for the probability of the event $E = \{\text{RB, BR}\}$ of the two picks being different colors, we would surmise that $p(E) = p(\text{RB}) + p(\text{BR}) = 23.\overline{3}\% + 23.\overline{3}\% = 46.\overline{6}\%$.

The next section is a further exploration of situations such as this one, in which not all outcomes in a sample space are equally likely.

Exercises for Section 5.3

1. A box contains six tickets: $\boxed{\text{A}}\boxed{\text{A}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\text{E}}$. You remove two tickets, one after the other. What is the probability that the first ticket is an A and the second is a B?
2. A box contains six tickets: $\boxed{\text{A}}\boxed{\text{A}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\text{E}}$. You remove two tickets, one after the other. What is the probability that both tickets are vowels?
3. In a shuffled 52-card deck, what is the probability that the top card is red and the bottom card is a heart?
4. A card is drawn off a 52-card deck. Let A be the event "The card is a heart." Let B be the event "The card is a queen." Are these two events independent or dependent?
5. Suppose A and B are events, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{6}$. Are A and B independent, dependent, or is there not enough information to say for sure?
6. Suppose A and B are events, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cup B) = \frac{2}{3}$. Are A and B independent, dependent, or is there not enough information to say for sure?
7. Say A and B are events with $P(A) = \frac{2}{3}$, $P(A|B) = \frac{3}{4}$, and $P(B|A) = \frac{1}{2}$. Find $p(B)$.
8. A box contains 2 red balls, 3 blue balls, and 1 green ball. You remove two balls, one after the other. Find the probability that both balls are red. Find the probability that both balls are blue. Find the probability that both balls have the same color.
9. A box contains 2 red balls, 3 black balls, and 4 white balls. One is removed, and then another is removed. What is the probability that no black balls were drawn?
10. A coin is flipped 5 times. What is the probability of all 5 tosses being tails? If there were more tails than heads, then what is the probability that all 5 tosses were tails?
11. A coin is flipped 5 times, and there are more tails than heads. What is the probability that the first flip was a tail?
12. Suppose events A and B are independent, $p(A) = \frac{1}{3}$, and $p(A \cup B) = \frac{2}{3}$. Find $p(B)$.
13. A 5-card hand is dealt from a shuffled 52-card deck. Exactly 2 of the cards in the hand are hearts. Find the probability that all the cards in the hand are red.
14. A 5-card hand is dealt from a shuffled 52-card deck. Exactly 2 of the cards in the hand are red. Find the probability that all the cards in the hand are hearts.

5.4 Probability Distributions and Probability Trees

Except for Example 5.14 on the previous page, we have, until now, assumed that any two outcomes in a sample space are equally likely to occur. This is reasonable in many situations, such as tossing an unbiased coin or dice, or dealing a hand from a shuffled deck.

But in reality, things are not always so uniform. Suppose the spots of a dice are hollowed out, and when tossed it is more likely to land with a lighter side up (one with more spots). Toss the dice once. The probabilities of the six outcomes in the sample space might be something like this:

$$\begin{array}{cccccccc} S & = & \{ & \text{⚀}, & \text{⚁}, & \text{⚂}, & \text{⚃}, & \text{⚄}, & \text{⚅} & \} \\ & & & 15\% & 15\% & 16\% & 16\% & 18\% & 20\% & \end{array}$$

(Of course it's unlikely the percentages would be whole numbers; this is just an illustration.) Note that the probabilities of all outcomes sum to 1:

$$p(\text{⚀}) + p(\text{⚁}) + p(\text{⚂}) + p(\text{⚃}) + p(\text{⚄}) + p(\text{⚅}) = 1,$$

because if tossed, the probability of its landing on one of its six faces is 100%. The probability of an event such as $E = \{\text{⚁}, \text{⚃}, \text{⚅}\}$ (*lands on even*) is

$$p(\text{⚁}) + p(\text{⚃}) + p(\text{⚅}) = 15 + 16 + 20 = 51\%.$$

Formula 5.1 does not apply here because the outcomes are not all equally likely. In fact it gives the incorrect probability $p(E) = \frac{|E|}{|S|} = \frac{3}{6} = 50\%$.

These ideas motivate the main definition of this section.

> **Definition 5.4** For an experiment with sample space $S = \{x_1, x_2, \ldots, x_n\}$, a **probability distribution** is a function p that assigns to each outcome $x_i \in S$ a probability $p(x_i)$ with $0 \le p(x_i) \le 1$, and for which
>
> $$p(x_1) + p(x_2) + \cdots + p(x_n) = 1.$$
>
> The **probability** $p(E)$ of an event $E \subseteq S$ is the sum of the probabilities of the elements of E.

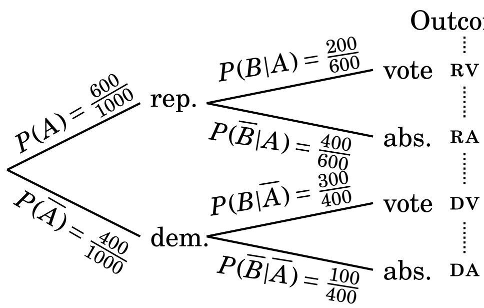
In the case where all outcomes are equally likely, any outcome $x_i \in S$ has probability $p(x_i) = \dfrac{1}{|S|}$. This is a probability distribution, by Definition 5.4. It is called the **uniform distribution** on S. For the uniform distribution we have the formula $p(E) = \dfrac{|E|}{|S|}$, but, as noted above, this may not hold for non-uniform probability distributions.

Example 5.15 A certain voting precinct has 1000 voters, 600 of whom are republicans and 400 of whom are democrats. In a recent election, 200 republicans voted and 300 democrats voted. You randomly select a member of the precinct and record whether they are republican or democrat, and whether or not they voted. Thus the sample space for the experiment is $S = \{\text{RV}, \text{RA}, \text{DV}, \text{DA}\}$, where RV means your selection was a republican who voted, whereas RA indicates a republican who abstained from voting, etc.

Find the probability distribution for S. Also, find the probability that your selection was a republican who voted or a democrat who didn't.

Solution. The chance that you picked a republican is $\frac{600}{1000} = 60\%$, and the chance that you picked a democrat is $\frac{400}{1000} = 40\%$. If you picked a republican, the conditional probability that this person voted is $\frac{200}{600}$, and the conditional probability that they didn't vote is $\frac{400}{600}$. If you picked a democrat, the conditional probability that this person voted is $\frac{300}{400}$, and the conditional probability that they didn't vote is $\frac{100}{400}$.

Here is the probability tree, where $A = \{\text{RV}, \text{RA}\}$ is the event of picking a republican and $B = \{\text{RV}, \text{DV}\}$ is the event of picking a voter.



$$P(A \cap B) = P(A) \cdot P(B|A) = \tfrac{6}{10} \cdot \tfrac{2}{6} = \tfrac{12}{60} = 20\%$$
$$P(A \cap \overline{B}) = P(A) \cdot P(\overline{B}|A) = \tfrac{6}{10} \cdot \tfrac{4}{6} = \tfrac{24}{60} = 40\%$$
$$P(\overline{A} \cap B) = P(\overline{A}) \cdot P(B|\overline{A}) = \tfrac{4}{10} \cdot \tfrac{3}{4} = \tfrac{12}{40} = 30\%$$
$$P(\overline{A} \cap \overline{B}) = P(\overline{A}) \cdot P(\overline{B}|\overline{A}) = \tfrac{4}{10} \cdot \tfrac{1}{4} = \tfrac{4}{40} = 10\%$$

Thus the probability distribution is

$$S = \left\{ \begin{array}{cccc} \text{RV}, & \text{RA}, & \text{DV}, & \text{DA} \\ 20\% & 40\% & 30\% & 10\% \end{array} \right\}$$

The probability that you picked a republican who voted or a democrat who didn't is $p\left(\{\text{RV}, \text{DA}\}\right) = p(\text{RV}) + p(\text{DA}) = 20\% + 10\% = 30\%$. ✍

If p is a probability distribution on a sample space S, then Definition 5.4 implies $p(S) = 1$ (because the probabilities of the elements of S sum to 1). Also, for mutually exclusive events $A, B \subseteq S$, we have $p(A \cup B) = p(A) + p(B)$, because Definition 5.4 says $p(A \cup B)$ is the sum probabilities of elements of A, plus those for B. And $S = A \cup \overline{A}$, so $1 = p(S) = p(A \cup \overline{A}) = p(A) + p(\overline{A})$.

This implies $p(A) = 1 - p(\overline{A})$ and $p(\overline{A}) = 1 - p(A)$. Therefore the formulas 2, 3 and 4 of Fact 5.2 (page 137) hold for arbitrary probability distributions, even though we derived them earlier only for uniform distributions.

Similar reasoning gives the following facts, which can be taken as a summary of all probability formulas in this chapter so far.

Probability Summary

Suppose p is a probability distribution for a sample space S of some experiment. Then the probability of an event $E = \{x_1, x_2, \dots x_k\} \subseteq S$ is $p(E) = p(x_1) + p(x_2) + \cdots + p(x_k)$. Thus $p(S) = 1$ and $p(\emptyset) = 0$.

If $A, B \subseteq S$ are arbitrary arbitrary events, then

$E = A \cup B$	is the event	"*A occurs* **or** *B occurs*,"
$E = A \cap B$	is the event	"*A occurs* **and** *B occurs*,"
$E = \overline{A}$	is the event	"*A* **does not** *occur*."

Events A and B are **mutually exclusive** if $A \cap B = \emptyset$, meaning $p(A \cap B) = p(\emptyset) = 0$, that is, A and B cannot both happen at the same time. In general:

1. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
2. $p(A \cup B) = p(A) + p(B)$ .... if A and B are mutually exclusive
3. $p\left(\overline{A}\right) = 1 - p(A)$
4. $p(A) = 1 - p\left(\overline{A}\right)$.

The **conditional probability** of A given B, denoted $p(A|B)$, is the probability of A, given that B has occurred. Events A and B are **independent** if $p(A|B) = p(A)$ and $p(B|A) = p(B)$, that is, if one happening does not change the probability that the other will happen.

5. $p(A|B) = \dfrac{p(A \cap B)}{p(B)}$
6. $p(B|A) = \dfrac{p(A \cap B)}{p(A)}$
7. $p(A \cap B) = p(A|B) \cdot p(B) = p(A) \cdot p(B|A)$
8. $p(A \cap B) = p(A) \cdot p(B)$ .... if A and B are independent

If p is the uniform distribution, then

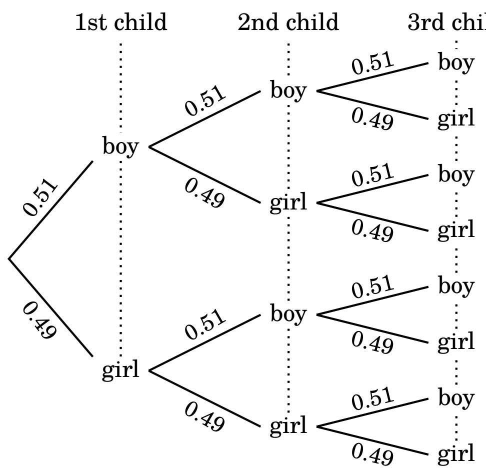
9. $p(A) = \dfrac{|A|}{|S|}$ .... if p is the uniform distribution.

At the beginning of Section 5.3 we calculated the probability that, for a family of three children, more girls than boys is just as likely as the oldest child being a boy. This was based on the assumption that there is a 50-50 chance of each child being a boy or a girl.

In reality, there is about a 51% chance of a child being born a boy, versus 49% for a girl. (Though the mortality rate for boys is higher, so this statistic is somewhat equalized in adulthood.) Let's revisit our question.

Example 5.16 Assume that there is 51% chance of a child being born a boy, versus 49% of being born a girl. For a family of three children, consider events A: *There are more girls than boys,* and B: *The oldest child is a boy*. Find $p(A)$ and $p(B)$.

Solution. The sample space is $S = \{\text{BBB}, \text{BBG}, \text{BGB}, \text{BGG}, \text{GBB}, \text{GBG}, \text{GGB}, \text{GGG}\}$. The following probability tree computes the probability of each outcome. (We assume that gender of births are independent, that is, the gender of one child does not influence the gender of the next child born.)



$p(\text{BBB}) = 0.51 \cdot 0.51 \cdot 0.51 = 0.132651$

$p(\text{BBG}) = 0.51 \cdot 0.51 \cdot 0.49 = 0.127449$

$p(\text{BGB}) = 0.51 \cdot 0.49 \cdot 0.51 = 0.127449$

$p(\text{BGG}) = 0.51 \cdot 0.49 \cdot 0.49 = 0.122451$

$p(\text{GBB}) = 0.49 \cdot 0.51 \cdot 0.51 = 0.127449$

$p(\text{GBG}) = 0.49 \cdot 0.51 \cdot 0.49 = 0.122451$

$p(\text{GGB}) = 0.49 \cdot 0.49 \cdot 0.51 = 0.122451$

$p(\text{GGG}) = 0.49 \cdot 0.49 \cdot 0.49 = 0.117649$

The probability $p(A)$ of more girls than boys is

$$\begin{aligned} p\big(\big(\text{BGG, GBG, GGB, GGG}\big\}\big) &= p(\text{BGG}) + p(\text{GBG}) + p(\text{GBG}) + p(\text{GGG}) \\ &= 0.122451 + 0.122451 + 0.122451 + 0.117649 \approx \mathbf{48.5\%}. \end{aligned}$$

The probability $p(B)$ of the oldest being a boy is

$$\begin{aligned} p\big(\big(\text{BBB, BBG, BGB, BGG}\big\}\big) &= p(\text{BGG}) + p(\text{GBG}) + p(\text{GBG}) + p(\text{GGG}) \\ &= 0.132651 + 127449 + 127449 + 0.122451 = 0.51 = \mathbf{51\%}. \end{aligned}$$

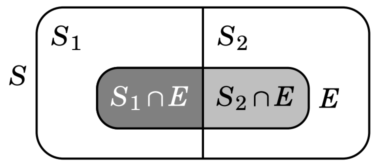
(This makes sense, as the chance of the firstborn being a boy is 51%.) ✍

Exercises for Section 5.4

1. There is a 40% chance of rain on Saturday and a 25% chance of rain on Sunday. What is the probability that it will rain on at least one day of the weekend? (You may assume that the events “Rain on Saturday” and “Rain on Sunday” are independent events.)
2. There is an 80% chance that there will be rain over the weekend, and a 50% chance of rain on Saturday. What is the chance of rain on Sunday? (You may assume that the events “Rain on Saturday” and “Rain on Sunday” are independent events.)
3. A club consists of 60 men and 40 women. To fairly choose a president and a secretary, names of all members are put into a hat and two names are drawn. The first name drawn is the president, and the second name drawn is the secretary. What is the probability that the president and the secretary have the same gender?
4. At a certain college, 40% of the students are male, and 60% are female. Also, 20% of the males are smokers, and 10% of the females are smokers. A student is chosen at random. What is the probability that the student is a male nonsmoker?
5. At a certain college, 30% of the students are freshmen. Also, 80% of the freshmen live on campus, while only 60% of the non-freshman students live on campus. A student is chosen at random. What is the probability that the student is a freshman who lives off campus?
6. Suppose events A and B are both independent *and* mutually exclusive. What can you say about $p(A)$ and $p(B)$?

5.5 Bayes' Formula

We are going to learn one final probability formula, *Bayes' formula*, named for its discoverer, Thomas Bayes (1702–1761). His formula gives an answer to the following question: Suppose a sample space S for an experiment is a union $S = S_1 \cup S_2$, with $S_1 \cap S_2 = \emptyset$, and $E \subseteq S$ is an event. If E occurs, then what is the probability that S_1 has occurred? That is, what is $p(S_1 \mid E)$?



A short computation gives the answer.

$$\begin{aligned}
p(S_1 \mid E) &= \frac{p(S_1 \cap E)}{p(E)} && \text{by formula 5 on page 152} \\
&= \frac{p(S_1) \cdot p(E \mid S_1)}{p(E)} && \text{by formula 7 on page 152} \\
&= \frac{p(S_1) \cdot p(E \mid S_1)}{p\big((S_1 \cap E) \cup (S_2 \cap E)\big)} && \text{as } E = (S_1 \cap E) \cup (S_2 \cap E) \\
&= \frac{p(S_1) \cdot p(E \mid S_1)}{p(S_1 \cap E) + p(S_2 \cap E)} && \text{because } S_1 \cap E \text{ and } S_2 \cap E \text{ are mutually exclusive} \\
&= \frac{p(S_1) \cdot p(E \mid S_1)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)}. && \text{by formula 7 on page 152}
\end{aligned}$$

The same steps give $p(S_2 \mid E) = \dfrac{p(S_2) \cdot p(E \mid S_2)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)}$.

Fact 5.4 Bayes' Formula
Suppose a sample space S for an experiment is a union $S = S_1 \cup S_2$, with $S_1 \cap S_2 = \emptyset$. Suppose also that $E \subseteq S$ is an event.

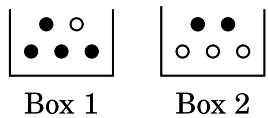
Then $p(S_1 \mid E) = \dfrac{p(S_1) \cdot p(E \mid S_1)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)}$

and $p(S_2 \mid E) = \dfrac{p(S_2) \cdot p(E \mid S_2)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)}$.

Though we will not use it here, we mention that Bayes' formula extends to situations in which S decomposes into more than two parts. If $S = S_1 \cup S_2 \cup \cdots \cup S_n$, and $S_i \cap S_j = \emptyset$ whenever $1 \le i < j \le n$, then for any S_i,

$$p(S_i \mid E) = \frac{p(S_i) \cdot p(E \mid S_i)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2) + \cdots + p(S_n) \cdot p(E \mid S_n)}. \tag{5.1}$$

Example 5.17 There are two boxes. Box 1 contains four black balls and one white ball. Box 2 contains two black balls and three white balls.



Someone chooses a box at random and then randomly takes a ball from it. The ball is white. What is the probability that the ball is from Box 1?

Solution. The sample space is $S = \{\,\text{1B},\ \text{1W},\ \text{2B},\ \text{2W}\,\}$, where the number refers to the box the selected ball came from, and the letter designates whether the ball is black or white.

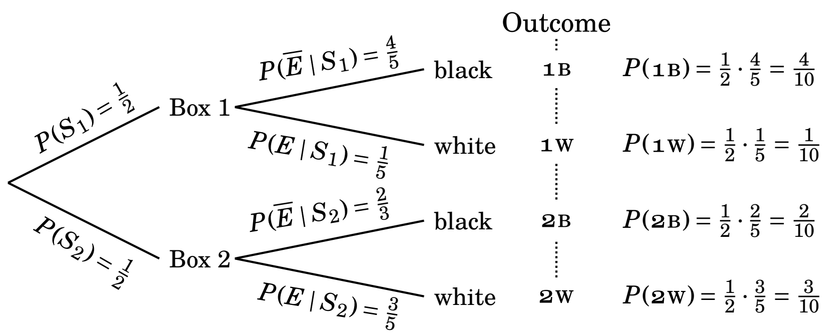
Let $S_1 = \{\text{1B},\ \text{1W}\}$ be the event S_1: *The ball came from Box* 1.
Let $S_2 = \{\text{2B},\ \text{2W}\}$ be the event S_2: *The ball came from Box* 2.
Let $E = \{\text{1W},\ \text{2W}\}$ be the event E: *The ball is white.*

The answer to the question is thus $p(S_1 \mid E)$. As $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, Bayes' formula applies, and it gives

$$\begin{aligned} p(S_1 \mid E) &= \frac{p(S_1)\cdot p(E \mid S_1)}{p(S_1)\cdot p(E \mid S_1) + p(S_2)\cdot p(E \mid S_2)} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{5}}{\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{3}{5}} = \frac{\frac{1}{10}}{\frac{1}{10}+\frac{3}{10}} = \frac{1}{4} = 25\%. \end{aligned}$$

So there's a 25% chance that the selected white ball came from Box 1. ✍

For full disclosure, let it be noted that we could bypass Bayes' formula and solve this problem with a probability tree. Note that $\overline{E} = \{\text{1B},\ \text{2B}\}$ is the event of a black ball being chosen. Consider the following probability tree.



Applying Formula 5 from page 152 to these figures, our answer is

$$p(S_1 \mid E) = \frac{p(S_1 \cap E)}{p(E)} = \frac{p(\{\text{1W}\})}{p(\{\text{1W},\ \text{2W}\})} = \frac{p(\text{1W})}{p(\text{1W}) + p(\text{2W})} = \frac{\frac{1}{10}}{\frac{1}{10}+\frac{3}{10}} = \frac{\frac{1}{10}}{\frac{4}{10}} = \frac{1}{4} = 25\%.$$

Exercises for Section 5.5

1. At a certain college, 40% of the students are male, and 60% are female. Also, 20% of the males are smokers, and 10% of the females are smokers. A student is chosen at random. If the student is a smoker, what is the probability that the student is female?

2. At a certain college, 30% of the students are freshmen. Also, 80% of the freshmen live on campus, while only 60% of the non-freshman students live on campus. A student is chosen at random. If the student lives on campus, what is the probability that the student is a freshman?

3. A jar contains 4 red balls and 5 white balls. A random ball is removed, and then another is removed. If the second ball was red, what is the probability that the first ball was red?