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A natural interpretation of fuzzy sets and fuzzy relations

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Abstract

We present a new and natural interpretation of fuzzy sets and fuzzy relations where the basic notions and operations have quite natural meanings. We interpret fuzzy sets and fuzzy relations in a cumulative Heyting valued model for intuitionistic set theory, and define the basic notions and operations naturally in the model. As far as fuzzy sets and fuzzy relations are considered as extensions of crisp sets and relations, this interpretation seems to be most natural.

In the interpretation the canonical embedding from the class of all sets into the model plays an important role. We distinguish generalized fuzzy sets, fuzzy subsets of crisp sets, and membership functions of fuzzy sets on crisp sets. Thus we present a foundation for developing a general theory of fuzzy sets where fuzzy subsets of different sets can be treated in a natural and uniform way.

Keywords: Fuzzy sets; Fuzzy relations; Fuzzy set operations; Heyting valued model; Intuitionistic set theory; Sheaf model

0. Introduction

The original concept of fuzzy sets in the pioneering paper of Zadeh [37] was introduced as an extension of usual (called *crisp*) sets, by enlarging the truth value set of 'grade of membership' from the two value set $\{0,1\}$ to the unit interval $[0,1]$ of real numbers.

Ordinary fuzzy sets are characterized by and mostly identified with mappings called 'membership functions' into $[0,1]$, which are extensions of characteristic functions of crisp sets. The basic relations between two fuzzy sets such as equality and inclusion and the basic fuzzy set operations such as intersection, union, and complement are defined by some equations or inequalities, which also hold between the characteristic functions of the corresponding crisp sets.

As for fuzzy relations, the situation is quite similar. Namely, either the basic operations such as composition and inverse or the basic properties such as reflexive, symmetric, transitive, etc., are defined by some equations as extensions of those in crisp relations.

But since $[0,1]$ has much more points than $\{0,1\}$, many kinds of extensions are definable. Indeed, various definitions have been proposed on the basic operations of fuzzy sets and fuzzy relations, and general notions such as t-norm have been considered and discussed (e.g., [1,7–9,24,38]). Sometimes the defining equations are presented with no mention

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of mathematical meanings, and various definitions are compared from the viewpoint of effectiveness to applications.

We interpret fuzzy sets and fuzzy relations in the model V^H introduced by the author [27], a Heyting valued model for intuitionistic set theory, where H is a complete Heyting algebra and is considered as the set of truth values in the model. V^H is a kind of the so-called sheaf model, cumulatively constructed by transfinite iteration of power sheaf construction over H.

In the first two sections we develop a basic theory of the model according to [27]. Although the basic set operations and relations in the model are not seen in the literature, the constructions are quite natural and the properties are very easy to verify, so all the proofs are omitted.

In the last two sections the fundamentals of the natural interpretation are described. In this interpretation the canonical embedding from standard universe (the class of all crisp sets) into the model, which assigns each crisp set to its 'check set', plays an important role. As far as fuzzy sets and relations are considered as extensions of crisp sets and relations, this interpretation seems to be most natural, for in the model the definitions and basic properties are almost the same as in crisp sets.

In Section 3 we present a general theory of extensions of fuzzy sets. We distinguish generalized fuzzy sets, fuzzy subsets of crisp sets, and membership functions of fuzzy subsets. Any set in the model is called an H-fuzzy set, and any subset (in the model) of the check set of a crisp set X is called an H-fuzzy subset of X. The membership function of an H-fuzzy set on X is defined as a mapping from X to H by using the canonical embedding. The membership functions are extensions of ordinary fuzzy sets [37] and instances of L-fuzzy sets [3].

For every crisp set X, there is a natural correspondence preserving order and basic set operations between H-fuzzy subsets of X and mappings from X to H. The order relation and the basic operations of mappings to H are pairwise defined. When $H=[0,1]$, the operations intersection and union correspond to the standard min and max, respectively, but the complement naturally corresponds to the intuitionistic (Gödel) negation, not to the ordinary (Zadeh) negation $x \mapsto 1-x$.

In Section 4 basic properties of H-fuzzy relations are dealt. An H-fuzzy relation between crisp sets is an H-fuzzy subset of the cartesian product. The max–min composition seems most natural as an extension of composition of crisp relations. Most of the defining properties for ordinary fuzzy equivalence and order relation are natural in the interpretation, but the ordinary definition of fuzzy linear order is weaker than the corresponding definition in the model.

Most of the literature on fuzzy sets treat only fuzzy subsets of a certain set called the universe. By the interpretation we can develop a natural theory of fuzzy sets without specifying the universe.

In the model for intuitionistic set theory, we can define many kinds of notions and operations. Hence besides those mentioned in this paper, various notions and operations between fuzzy sets and fuzzy relations can be treated in a natural way. Further study on the natural interpretation of fuzzy relations, fuzzy mappings, and extension principles will appear in a forthcoming paper [28].

Before closing this section, we consider the relation with some preceding researches. In the following 'the internal logic' means the logic in the model.

The Heyting valued model in the present paper has its origin in [27], and is very similar to the sheaf models in [10,11,23,29,32,33,35] (where the Heyting algebra is denoted by Ω). These models are extensions of (the so-called Scott–Solovay) Boolean valued models for classical set theory. Takuti and Titani [30,31,34] also use similar sheaf models for studying fuzzy logic and fuzzy set theory. But in these papers the internal logic (intuitionistic fuzzy logic; IFL in [30] or FL in [34]) has extra axioms (and logical symbols in [34]) to express properties of the unit interval $[0,1]$, and in their sheaf model the valuation and interpretation are a little different from ours. Moreover, in these works the operations and relations such as set inclusion, basic set operations, and compositions of relations etc. are not treated in connection with their membership functions. Hence though the model is a little similar, our interpretation is original and new.

As far as we know, the only precedent of our interpretation appears in Kodera [23], where the interpretation is applied only to elements of a group.

Zhang [39] gives a generalization of Boolean valued model and claims that this model can give a unified treatment of fuzzy set theory and Boolean valued set theory. But it is merely a rough sketch of the model. Weidner [36] presents a formal axiomatic system for fuzzy set theory and shows that Boolean valued universe can be interpreted to form a model of the system. In both papers the internal logic is the classical logic and there are no precise arguments on the interpretation of basic notions and operations of fuzzy sets and fuzzy relations, and we do not know how to calculate membership functions of operations of fuzzy sets or relations.

Gottwald [6] presents another kind of cumulative model (cf. [5,7]). Here the internal logic is a version of the Lukasiewicz logic and have two conjunctions and two disjunctions, and the definition of the truth values of atomic formulae is simpler than ours. Although one can develop set theory in this model to some extent, it looks too complicated to study more, for many notions have two kind of definitions according to the two kind of logical connectives. So it seems to reflect no way that fuzzy sets are usually used (cf. [22]).

The internal logic of our model is the intuitionistic logic with identity and existence, introduced by Scott [26]. The semantics for this logic using Heyting valued sets are studied in [2,12,4]. Related to this logic and its extensions, Höhle [18,20,21] investigates category theoretical properties of M-valued sets and sheaves over M, where M is a GL-monoid or GL-quantale or a GL- algebra (cf. also [13–17,19,25]). Many results of Higgs and Fourman-Scott (cf. [2]) are instances of theorems in Höhle's papers when M is a complete Heyting algebra. As a category the model V^H of the present paper is equivalent to the category of Sheaves over H and the category of H-valued sets [27]. But our work is one of set-theoretic approaches, and category theoretic approach is beyond the scope of the present paper [22]. The relation with Höhle's work might be an interesting subject of future study.

1. Heyting valued model for intuitionistic set theory

Let H be a fixed complete Heyting algebra. We adopt the usual notations such as $\wedge, \vee, \bigwedge, \bigvee, \to, \neg, \mathbf{0}, \mathbf{1}, \leqslant$. For complete Heyting algebras, see e.g. Fourman–Scott [2].

Following Shimoda [27], we construct an H-valued model for the *extended intuitionistic set theory*, which is a first order intuitionistic logic with predicates $\in, =$ and E together with axioms of intuitionistic set theory. The unary predicate E is called an *existence predicate*, for Eτ means 'τ exists'. We assume that $\in$ and $=$ are *strict*, that is, in addition to the usual axioms of intuitionistic logic, we have two strictness axioms: $x \in y \to \mathrm{E}x \wedge \mathrm{E}y$, and $x = y \to \mathrm{E}x \wedge \mathrm{E}y$. The theory of intuitionistic logic with existence predicate is developed in Scott [26]. For the axioms of intuitionistic set theory, see e.g. Grayson [11] or Titani [35].

Let V be the class of all usual sets and On be the class of all ordinals.

Definition 1.1. The model V^H is constructed as follows.

For every ordinal α, V^H_α is defined by induction:

$$V^H_0 = \phi, \quad V^H_\alpha = \bigcup_{\beta<\alpha} V^H_\beta \quad (\text{if } \alpha \text{ is a limit ordinal}),$$

$$V^H_{\alpha+1} = \{u = \langle |u|, Eu\rangle; |u| : \mathscr{D}u \to H,\ \mathscr{D}u \subseteq V^H_\alpha,\ Eu \in H,\ |u|(x) \leqslant Eu \wedge Ex \text{ for all } x \in \mathscr{D}u\}.$$

Then the *H-valued model* is

$$V^H = \bigcup_{\alpha \in On} V^H_\alpha.$$

An element of V^H is called a *set in* V^H. We usually identify $|u|$ with u.

If we write a formula as $\varphi(a_1, \ldots, a_n)$, we mean that all the free variables of φ are among $\{a_1, \ldots, a_n\}$. In particular, φ is always a sentence. A *formula* (resp. *sentence*) *of* V^H is a formula (resp. sentence) of intuitionistic set theory with constants from (names of) elements of V^H.

Definition 1.2. For a sentence φ of V^H, the *Heyting value* $\|\varphi\| \in H$ is defined as follows. Let $u, v \in V^H$, φ and ψ be sentences of V^H, and $\varphi(a)$ be a formula of V^H.

For atomic formulae,

$$\|\mathrm{E}u\| = Eu \quad \text{and by simultaneous induction,}$$

$$\|u \in v\| = \bigvee_{y \in \mathscr{D}v} (v(y) \wedge \|u = y\|) \quad \text{and}$$

$$\|u = v\| = \bigwedge_{x \in \mathscr{D}u} (u(x) \to \|x \in v\|) \wedge \bigwedge_{y \in \mathscr{D}v} (v(y) \to \|y \in u\|) \wedge Eu \wedge Ev.$$

For compounded formulae, by induction (on the number of logical symbols),

$$\|\varphi \wedge \psi\| = \|\varphi\| \wedge \|\psi\|, \quad \|\varphi \vee \psi\| = \|\varphi\| \vee \|\psi\|,$$

$$\|\varphi \to \psi\| = \|\varphi\| \to \|\psi\|, \quad \|\neg \varphi\| = \neg \|\varphi\|,$$

$$\|\forall x \varphi(x)\| = \bigwedge_{u \in V^H} (Eu \to \|\varphi(x)\|) \quad \text{and}$$

$$\|\exists x \varphi(x)\| = \bigvee_{u \in V^H} (Eu \wedge \|\varphi(x)\|).$$

Although we use the same symbols for logical connectives and operations of a Heyting algebra, the meanings will be clear from the context. A sentence φ of V^H is called *valid in* V^H iff $\|\varphi\| = \mathbf{1}$. It is easy to verify that every sentence provable in the extended intuitionistic logic is valid in V^H. The strictness axioms are also valid by the following lemma.

Lemma 1.1. *Let u, v be sets in V^H.*

(1) *$\|u \in v\| \leqslant Eu \wedge Ev$, and $\|u = v\| \leqslant Eu \wedge Ev$.*
(2) *$\|u = u\| = Eu$, and $u(x) \leqslant \|x \in u\|$ for all $x \in \mathscr{D}u$.*

Equality axioms for every formula is also valid in the model.

Lemma 1.2. *For every formula $\varphi(a_1, \ldots, a_n)$ of V^H,*

$$\|u_1 = v_1\| \wedge \cdots \|u_n = v_n\| \wedge \|\varphi(u_1, \ldots, u_n)\| \leqslant \|\varphi(v_1, \ldots, v_n)\|$$

for all $u_1, \ldots, u_n, v_1, \ldots, v_n \in V^H$.

We abbreviate $\forall x (x \in u \to \varphi(x))$ to $(\forall x \in u)\varphi(x)$ and $\exists x (x \in u \wedge \varphi(x))$ to $(\exists x \in u)\varphi(x)$, respectively, as usual. Quantifiers of the above form is called *bounded*, and a *bounded formula* is a formula in which every quantifier is bounded. The following equations are useful for various calculations.

Proposition 1.3. *Let $\varphi(a)$ be a formula of V^H and $u \in V^H$.*

(1) $\|(\forall x \in u)\varphi(x)\| = \bigwedge_{x \in \mathscr{D}u} (u(x) \to \|\varphi(x)\|) = \bigwedge_{x \in \mathscr{D}u} (\|x \in u\| \to \|\varphi(x)\|).$
(2) $\|(\exists x \in u)\varphi(x)\| = \bigvee_{x \in \mathscr{D}u} (u(x) \wedge \|\varphi(x)\|) = \bigvee_{x \in \mathscr{D}u} (\|x \in u\| \wedge \|\varphi(x)\|).$

Hence for $u, v \in V^H$,

$$\|u \subseteq v\| = \bigwedge_{x \in \mathscr{D}u} (u(x) \to \|x \in v\|) = \bigwedge_{x \in \mathscr{D}u} (\|x \in u\| \to \|x \in v\|).$$

If $\|u \subseteq v\| = \mathbf{1}$, we say u is a *subset of v in V^H* and write $u \sqsubseteq v$. If $u \sqsubseteq v$ and $v \sqsubseteq u$, we say u and v are *similar* and denote it as $u \sim v$, and when $u \sim v$ and $Eu = Ev$, u and v are said to be *equivalent* and denoted by $u \approx v$.

The inclusion relation $\sqsubseteq$ is a preorder, and the similarity relation $\sim$ is the equivalence relation induced from $\sqsubseteq$. The relation $\approx$ is another equivalence relation stronger than $\sim$.

Lemma 1.4. *Let $u, v \in V^H$.*

(1) *$\|u = v\| = \|u \subseteq v\| \wedge \|v \subseteq u\| \wedge Eu \wedge Ev$.*
(2) *$u \sqsubseteq v$ iff $\|x \in u\| \leqslant \|x \in v\|$ for all $x \in V^H$.*
(3) *$u \sim v$ iff $\|x \in u\| = \|x \in v\|$ for all $x \in V^H$.*
(4) *$u \approx v$ iff $\|u = v\| = Eu = Ev$.*

Proposition 1.5. *Let $\varphi(a)$ be a formula of V^H and u, v be sets in V^H. If u and v are equivalent, then $\|\varphi(u)\| = \|\varphi(v)\|$.*

Hence equivalent elements are substitutive in any sentence. The proof is done by induction on the complexity of formula, using the equality axioms.

A set u in V^H is *extensional* iff $\|x = y\| \wedge u(y) \leqslant u(y)$ for all $x, y \in \mathscr{D}u$. Then u is extensional iff $u(x) = \|x \in u\|$ for all $x \in \mathscr{D}u$ iff there exists a formula $\varphi(a)$ of V^H such that $u(x) = \|\varphi(x)\|$ for all $x \in \mathscr{D}u$.

The *canonical embedding* of the standard universe V into the model V^H is defined as follows. For $x \in V$ define $\check{x} \in V^H$ recursively by

$$\mathscr{D}(\check{x}) = \{\check{y};\, y \in x\}, \quad E\check{x} = \mathbf{1}, \quad \check{x} : \check{y} \mapsto \mathbf{1}.$$

We call $\check{x}$ the *check set* of x.

Note that the check set $\check{\phi}$ of the empty set ϕ in V is an empty set in V^H, that is, $\|\exists x(x\in\check{\phi})\|=\mathbf{0}$, and $\|\check{\phi}\subseteq u\|=\mathbf{1}$ for all $u\in V^H$. We write ϕ instead of $\check{\phi}$ if there is no fear of confusion.

Proposition 1.6. *Suppose $\varphi(a_1,\dots,a_n)$ is a bounded formula of V^H and $x_1,\dots,x_n\in V$. Then the following hold.*

$\varphi(x_1,\dots,x_n)$ *holds iff* $\|\varphi(\check{x}_1,\dots,\check{x}_n)\|=\mathbf{1}$, *and*

$\neg\varphi(x_1,\dots,x_n)$ *holds iff* $\|\varphi(\check{x}_1,\dots,\check{x}_n)\|=\mathbf{0}$.

We shall define some basic constructions of sets in the model. First we define pair, couple (ordered pair), (cartesian) product, union, and power set in V^H.

Definition 1.3. For $u,v\in V^H$, $\{u,v\}^H$, $\langle u,v\rangle^H$, $u\overset{H}{\times}v$, $\bigcup^H u$, $\mathscr{P}^H u$ are defined in the following way.

(1) $\mathscr{D}\{u,v\}^H=\{u,v\}$, $E\{u,v\}^H=Eu\wedge Ev$, and $\{u,v\}^H:x\mapsto\mathbf{1}$.
(2) $\langle u,v\rangle^H=\{\{u,u\}^H,\{u,v\}^H\}^H$.
(3) $\mathscr{D}(u\overset{H}{\times}v)=\{\langle x,y\rangle^H;x\in\mathscr{D}u,y\in\mathscr{D}v\}$, $E(u\overset{H}{\times}v)=Eu\wedge Ev$, and $u\overset{H}{\times}v:\langle x,y\rangle^H\mapsto u(x)\wedge v(y)$.
(4) $\mathscr{D}(\bigcup^H u)=\bigcup_{y\in\mathscr{D}u}\mathscr{D}y$, $E(\bigcup^H u)=Eu$, and $\bigcup^H u:x\mapsto\|(\exists y\in u)(x\in y)\|$.
(5) $\mathscr{D}(\mathscr{P}^H u)=\{x\in V^H;\mathscr{D}x=\mathscr{D}u\}$, $E(\mathscr{P}^H u)=Eu$, and $\mathscr{P}^H u:x\mapsto\|x\subseteq u\|\wedge Ex\wedge Eu$.

Pairs, couples, and unions are extensional. The commas in couples and the superscripts are sometimes omitted when the meaning is clear from the context. For example, $\langle x,y\rangle^H$ is abbreviated to $\langle xy\rangle^H$ or $\langle xy\rangle$.

Lemma 1.7. *Let $x,y,u,v\in V^H$.*

(1) $\|x\in\{u,v\}^H\|=(\|x=u\|\vee\|x=v\|)\wedge Eu\wedge Ev$.
(2) $\|\langle uv\rangle^H=\langle xy\rangle^H\|=\|u=x\|\wedge\|v=y\|$.
(3) $\|\langle xy\rangle^H\in u\overset{H}{\times}v\|=\|x\in u\|\wedge\|y\in v\|$.
(4) $\|x\in\bigcup^H u\|=\|(\exists y\in u)(x\in y)\|$.
(5) $\|x\in\mathscr{P}^H u\|=\|x\subseteq u\|\wedge Ex\wedge Eu$.

Lemma 1.8. *For all $x,y,u,v\in V$, the following hold.*

(1) $\{\check{x},\check{y}\}^H=\{x,y\}\check{}$.
(2) $\langle\check{x}\check{y}\rangle^H=\langle xy\rangle\check{}$.
(3) $\check{u}\overset{H}{\times}\check{v}=(u\times v)\check{}$.

The following proposition indicates that V^H is a model for intuitionistic set theory. The proof is quite similar to that in [10] or [35].

Proposition 1.9. *All axioms of intuitionistic set theory are valid in V^H.*

Now we present the definitions of basic set operations in the model.

Definition 1.4. For $u,v\in V^H$, define the *intersection*, *union*, and *difference* of u and v as follows.

(1) $\mathscr{D}(u\cap^H v)=\mathscr{D}u\cup\mathscr{D}v$, $E(u\cap^H v)=Eu\wedge Ev$, and $u\cap^H v:x\mapsto\|x\in u\|\wedge\|x\in v\|$.
(2) $\mathscr{D}(u\cup^H v)=\mathscr{D}u\cup\mathscr{D}v$, $E(u\cup^H v)=Eu\vee Ev$, and $u\cup^H v:x\mapsto\|x\in u\|\vee\|x\in v\|$.
(3) $\mathscr{D}(u\backslash^H v)=\mathscr{D}u\cup\mathscr{D}v$, $E(u\backslash^H v)=Eu\vee Ev$, and $u\backslash^H v:x\mapsto\|x\in u\|\wedge\neg\|x\in v\|$.

Intersections, unions, and differences are also extensional. We usually omit the superscripts if there is no fear of confusion, and denote by $u\cap v$, $u\cup v$, and $u\backslash v$.

Note that $\bigcup^H\{u,v\}$ and $u\cup v$ are similar but not equivalent unless $Eu=Ev$.

Lemma 1.10. *For all $x,u,v\in V^H$, the following hold*:

(1) $\|x\in u\cap v\|=\|x\in u\|\wedge\|x\in v\|$.
(2) $\|x\in u\cup v\|=\|x\in u\|\vee\|x\in v\|$.
(3) $\|x\in u\backslash v\|=\|x\in u\|\wedge\neg\|x\in v\|$.

The following lemma shows the lattice theoretic properties of the basic set operations.

Lemma 1.11. *For all $u,v,z\in V^H$, the following hold*:

(1) *$u\cap v\sqsubseteq u$, $u\cap v\sqsubseteq v$, and if $z\sqsubseteq u$ and $z\sqsubseteq v$, then $z\subseteq u\cap v$.*
(2) *$u\sqsubseteq u\cup v$, $v\sqsubseteq u\cup v$, and if $u\sqsubseteq z$ and $v\sqsubseteq z$, then $u\cup v\subseteq z$.*
(3) *$u\backslash v\sqsubseteq u$, $(u\backslash v)\cap v\sim\phi$, and if $z\sqsubseteq u$ and $z\cap v\sim\phi$, then $z\sqsubseteq u\backslash v$.*

Hence considering the equivalence classes with respect to $\sim$, $u\cap v$ and $u\cup v$ play the role of the greatest lower bound and the least upper bound of u and v, respectively, and $u\backslash v$ works as the relative complement of v to u.

Definition 1.5. For a family $\{u_\lambda\}_{\lambda\in\Lambda}$ of sets in V^H, the *intersection* $\bigcap_{\lambda\in\Lambda} u_\lambda$ and the *union* $\bigcup_{\lambda\in\Lambda} u_\lambda$ are defined as follows:

(1) $\mathscr{D}(\bigcap_{\lambda\in\Lambda} u_\lambda) = \bigcup_{\lambda\in\Lambda} \mathscr{D}u_\lambda$,
$E(\bigcap_{\lambda\in\Lambda} u_\lambda) = \bigvee_{\lambda\in\Lambda} Eu_\lambda$, and
$\bigcap_{\lambda\in\Lambda} u_\lambda : x \mapsto \bigwedge_{\lambda\in\Lambda} \|x \in u_\lambda\|$.
(2) $\mathscr{D}(\bigcup_{\lambda\in\Lambda} u_\lambda) = \bigcup_{\lambda\in\Lambda} \mathscr{D}u_\lambda$,
$E(\bigcup_{\lambda\in\Lambda} u_\lambda) = \bigvee_{\lambda\in\Lambda} Eu_\lambda$, and
$\bigcup_{\lambda\in\Lambda} u_\lambda : x \mapsto \bigvee_{\lambda\in\Lambda} \|x \in u_\lambda\|$.

The following are extensions of the previous two lemmas.

Lemma 1.12. *Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be a family of sets in V^H and x be a set in V^H.*

(1) $\|x \in \bigcap_{\lambda\in\Lambda} u_\lambda\| = \bigwedge_{\lambda\in\Lambda} \|x \in u_\lambda\|$.
(2) $\|x \in \bigcup_{\lambda\in\Lambda} u_\lambda\| = \bigvee_{\lambda\in\Lambda} \|x \in u_\lambda\|$.

Lemma 1.13. *Let $u_\lambda (\lambda\in\Lambda)$ and z be sets in V^H.*

(1) *$\bigcap_{\lambda\in\Lambda} u_\lambda \sqsubseteq u_\lambda$ for every $\lambda\in\Lambda$, and if $z \sqsubseteq u_\lambda$ for every $\lambda\in\Lambda$, then $z \sqsubseteq \bigcap_{\lambda\in\Lambda} u_\lambda$.*
(2) *$u_\lambda \sqsubseteq \bigcup_{\lambda\in\Lambda} u_\lambda$ for every $\lambda\in\Lambda$, and if $u_\lambda \sqsubseteq z$ for every $\lambda\in\Lambda$, then $\bigcup_{\lambda\in\Lambda} u_\lambda \sqsubseteq z$.*

2. Relations in the model

A *relation in V^H* is a subset of a cartesian product of two sets in V^H. For $R, u, v \in V^H$, R is a *relation from u to v* in V^H iff $R \sqsubseteq u \overset{H}{\times} v$. A *relation on u* is a relation from u to u. We often write xRy instead of $\langle xy\rangle \in R$ as usual.

Definition 2.1. For $u \in V^H$, the *identity relation I_u on u* is defined by

$$\mathscr{D}(I_u) = \{\langle xx\rangle; x \in \mathscr{D}u\}, \quad E(I_u) = Eu,$$
$$\text{and} \quad I_u : \langle xx\rangle \mapsto u(x).$$

I_u is also called the *equality relation on u*.

Lemma 2.1. *Let $u \in V^H$.*

(1) *$\|xI_u y\| = \|x \in u\| \wedge \|x = y\|$ for all $x, y \in V^H$.*
(2) *$\|z \in I_u\| = \|(\exists x \in u)(z = \langle xx\rangle)\|$ for all $z \in V^H$.*
(3) *$I_u \sqsubseteq u \times u$, that is, I_u is a relation on u.*

The definition of the identity relation is not unique, for another set in V^H similar to I_u plays the same role. Indeed, let $i_u \in V^H$ be defined by

$$\mathscr{D}(i_u) = \{\langle xy\rangle; x, y \in \mathscr{D}u\}, \quad E(i_u) = Eu,$$
$$\text{and} \quad i_u : \langle xy\rangle \mapsto \|x = y\|.$$

Then it is easy to show that $I_u \approx i_u$, and the lemma also holds if i_u is substituted for I_u.

Definition 2.2. For $R, S \in V^H$, the *composition $S \circ R$ in V^H* is defined by

$$\mathscr{D}(S \circ R) = \{\langle xz\rangle; x, z \in V^H_\beta\},$$
$$E(S \circ R) = ER \wedge ES, \quad \text{and}$$
$$S \circ R : \langle xz\rangle \mapsto \|\exists y(xRy \wedge ySz)\|,$$

where β is an ordinal which satisfies $\mathscr{D}R \cup \mathscr{D}S \subseteq V^H_\beta$.

The definition is not affected by the choice of the ordinal β. Note that the order of the composition is the same as for usual functions, and inverse to that of fuzzy relations in most fuzzy literature.

Lemma 2.2. *Let $R, S, u, v, w \in V^H$. Then each of the following holds*:

(1) *$\|x(S \circ R)z\| = \|\exists y(xRy \wedge ySz)\|$ for all $x, z \in V^H$.*
(2) *$\|t \in S \circ R\| = \|\exists x\exists y\exists z(t = \langle xz\rangle \wedge xRy \wedge ySz)\|$ for all $t \in V^H$.*
(3) *If $R \sqsubseteq u \times v$ and $S \sqsubseteq v \times w$, then $S \circ R \sqsubseteq u \times w$.*

We have the same properties as in usual sets.

Proposition 2.3. *Let $u, v, R, S, T \in V^H$ and $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of sets in V^H.*

(1) *If $R \sqsubseteq u \times v$, then $R \circ I_u \sim R \sim I_v \circ R$.*
(2) *$(T \circ S) \circ R \approx T \circ (S \circ R)$.*
(3) *$T \circ (\bigcap_{\lambda\in\Lambda} S_\lambda) \sqsubseteq \bigcap_{\lambda\in\Lambda} (T \circ S_\lambda)$, $(\bigcap_{\lambda\in\Lambda} S_\lambda) \circ R \sqsubseteq \bigcap_{\lambda\in\Lambda} (S_\lambda \circ R)$.*

(4) $T\circ(S\cap R)\sqsubseteq(T\circ S)\cap(T\circ R)$,
$(T\cap S)\circ R\sqsubseteq(T\circ R)\cap(S\circ R)$.
(5) $T\circ(\bigcup_{\lambda\in\Lambda}S_\lambda)\approx\bigcup_{\lambda\in\Lambda}(T\circ S_\lambda)$,
$(\bigcup_{\lambda\in\Lambda}S_\lambda)\circ R\approx\bigcup_{\lambda\in\Lambda}(S_\lambda\circ R)$.
(6) $T\circ(S\cup R)\approx(T\circ S)\cup(T\circ R)$,
$(T\cup S)\circ R\approx(T\circ R)\cup(S\circ R)$.
(7) *If* $R\sqsubseteq S$, *then* $T\circ R\sqsubseteq T\circ S$ *and* $R\circ T\sqsubseteq S\circ T$.

Note that the inverses of the inclusion relations in (3) and (4) do not hold in general.

Definition 2.3. For $R\in V^H$, the *inverse relation* R^{-1} in V^H is defined by

$$\mathscr{D}(R^{-1})=\{\langle xy\rangle;x,y\in V_\beta^H\},$$

$$E(R^{-1})=ER \quad\text{and}\quad R^{-1}:\langle xy\rangle\mapsto\|yRx\|,$$

where β is an ordinal which satisfies $\mathscr{D}R\subseteq V_\beta^H$.

The definition is not affected by the choice of the ordinal β.

Lemma 2.4. *For* $u,v,R\in V^H$, *each of the following holds*:

(1) $\|xR^{-1}y\|=\|yRx\|$ *for all* $x,y\in V^H$.
(2) $\|z\in R^{-1}\|=\|\exists x\exists y(z=\langle xy\rangle\ \wedge\ yRx)\|$ *for all* $z\in V^H$.
(3) *If* $R\sqsubseteq u\times v$, *then* $R^{-1}\sqsubseteq v\times u$.
(4) $(u\times v)^{-1}\approx v\times u$.

Proposition 2.5. *For all* $u,v,R,S,\{R_\lambda\}_{\lambda\in\Lambda}$ *in* V^H, *the following hold:*

(1) $(R^{-1})^{-1}\sqsubseteq R$.
If $R\sqsubseteq u\times v$, *then* $(R^{-1})^{-1}\approx R$.
(2) $(\bigcap_{\lambda\in\Lambda}R_\lambda)^{-1}\approx\bigcap_{\lambda\in\Lambda}R_\lambda^{-1}$,
$(\bigcup_{\lambda\in\Lambda}R_\lambda)^{-1}\approx\bigcup_{\lambda\in\Lambda}R_\lambda^{-1}$.
(3) $(R\cap S)^{-1}\approx R^{-1}\cap S^{-1}$, $(R\cup S)^{-1}\approx R^{-1}\cup S^{-1}$.
(4) $(R\backslash S)^{-1}\approx R^{-1}\backslash S^{-1}$,
$((u\times v)\backslash R)^{-1}\approx(v\times u)\backslash R^{-1}$.
(5) $(S\circ R)^{-1}\approx R^{-1}\circ S^{-1}$.
(6) *If* $R\sqsubseteq S$, *then* $R^{-1}\sqsubseteq S^{-1}$.

Some properties on relations on a set in V^H are defined in the same way as in V. For instance, a relation R on u in V^H is *reflexive* iff $\|R$ is reflexive$\|=\mathbf{1}$. More explicitly, we define as follows.

Definition 2.4. Let $u\in V^H$ and R be a relation on u in V^H.

(1) R is *reflexive* (*in* V^H) iff $\|(\forall x\in u)(xRx)\|=\mathbf{1}$.
(2) R is *symmetric* (*in* V^H)
iff $\|\forall x\forall y(xRy\to yRx)\|=\mathbf{1}$.
(3) R is *transitive* (*in* V^H)
iff $\|\forall x\forall y\forall z(xRy\wedge yRz\to xRz)\|=\mathbf{1}$.
(4) R is *antisymmetric* (*in* V^H)
iff $\|\forall x\forall y(xRy\wedge yRx\to x=y)\|=\mathbf{1}$.
(5) R is *connected* (*in* V^H)
iff $\|(\forall x\in u)(\forall y\in u)(xRy\vee yRx)\|=\mathbf{1}$.

In general, a relation on a set is an *equivalence relation* if it is reflexive, symmetric, and transitive, and is a *partial order* (or an *order relation*) if it is reflexive, antisymmetric, and transitive. A connected partial order is called a *linear order* or a *total order*. In the same way *equivalence relation*, *partial order*, and *linear order in* V^H are defined.

Lemma 2.6. *For a relation* R *on* $u\in V^H$, *the following hold:*

(1) *R is reflexive iff* $\|x\in u\|\leqslant\|xRx\|$ *for all* $x\in V^H$.
(2) *R is symmetric*
iff $\|xRy\|\leqslant\|yRx\|$ *for all* $x,y\in V^H$
iff $\|xRy\|=\|yRx\|$ *for all* $x,y\in V^H$.
(3) *R is transitive*
iff $\|xRy\|\wedge\|yRz\|\leqslant\|xRz\|$ *for all* $x,y,z\in V^H$.
(4) *R is antisymmetric*
iff $\|xRy\|\wedge\|yRx\|\leqslant\|x=y\|$ *for all* $x,y\in V^H$.
(5) *R is connected*
iff $\|x\in u\|\wedge\|y\in u\|\leqslant\|xRy\|\vee\|yRx\|$ *for all* $x,y\in V^H$.

Using identity, composition, and inverse relations, equivalent conditions of the above properties are expressed.

Lemma 2.7. *Let* R *be a relation* R *on* $u\in V^H$.

(1) *R is reflexive iff* $I_u\sqsubseteq R$.
(2) *R is symmetric iff* $R^{-1}\sqsubseteq R$ *iff* $R^{-1}\approx R$.
(3) *R is transitive iff* $R\circ R\sqsubseteq R$.
(4) *R is antisymmetric iff* $R\cap R^{-1}\sqsubseteq I_u$.
(5) *R is connected iff* $u\times u\sqsubseteq R\cup R^{-1}$
iff $u\times u\sim R\cup R^{-1}$.

3. Natural interpretation of fuzzy sets

Let X be a fixed usual (crisp) set.

Definition 3.1. We call every set in V^H an *H-fuzzy set*.

For an H-fuzzy set A, the *membership function of A on X* is the mapping

$$\mu_A = \mu_A^X : X \to H, \quad x \mapsto \|\check{x} \in A\|.$$

An *H-fuzzy subset* of X is a subset of $\check{X}$ in V^H, that is, for $A \in V^H$,

A is an H-fuzzy subset of X iff $A \sqsubseteq \check{X}$.

This is an extension of the ordinary definition of fuzzy sets. Ordinarily a fuzzy set on X is identified with its membership function from X to $[0,1]$. An ordinary fuzzy set on X is a membership function of an H-fuzzy subset of X, when $H = [0,1]$.

The superscript in the notation of a membership function is often omitted if the 'underlying set' is clear by the context. In this section we consider only the membership functions of H-fuzzy sets on the fixed set X.

Proposition 3.1. *Every mapping from X to H is the membership function of some H-fuzzy subset of X.*

Proof. For a mapping $\mu : X \to H$, define $A \in V^H$ by

$$\mathscr{D}\mathscr{A} = \{\check{x}; x \in X\}, \quad EA = \bigvee_{x \in X} \mu(x), \quad \text{and}$$
$$A : \check{x} \mapsto \mu(x).$$

Since $\mathscr{D}\mathscr{A} = \mathscr{D}\check{X}$, obviously $A \sqsubseteq \check{X}$, that is, A is an H-fuzzy subset of X. Then for all $x \in X$,

$$\mu_A(x) = \|\check{x} \in A\| = \bigvee_{u \in \mathscr{D}\mathscr{A}} (A(u) \wedge \|\check{x} = u\|)$$
$$= \bigvee_{y \in X} (A(\check{y}) \wedge \|\check{x} = \check{y}\|) = A(\check{x}) = \mu(x).$$

Hence $\mu = \mu_A$. □

There is a natural correspondence between H-fuzzy subsets of X and mappings from X to H. It will be shown that the correspondence $A \mapsto \mu_A$ preserves not only the order (of inclusion), but also the set operations.

The order relation and basic set operations of mappings into H are defined pointwise as usual.

Definition 3.2. Let μ, ν be mappings from X to H.

(1) $\mu \leqslant \nu$ iff $\mu(x) \leqslant \nu(x)$ for all $x \in X$.
(2) The mappings $\mu \wedge \nu, \mu \vee \nu, \neg\mu : X \to H$ are defined by

$$\mu \wedge \nu : x \mapsto \mu(x) \wedge \nu(x),$$
$$\mu \vee \nu : x \mapsto \mu(x) \vee \nu(x) \quad \text{and}$$
$$\neg\mu : x \mapsto \neg\mu(x)$$

for all $x \in X$.

In case $H = [0,1]$, the values of these operations are as follows:

$$(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\},$$
$$(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\} \quad \text{and}$$

$$(\neg\mu)(x) = \begin{cases} 1 & \text{if } \mu(x) = 0, \\ 0 & \text{if } \mu(x) > 0. \end{cases}$$

Theorem 1. *Let A, B be H-fuzzy sets, that is, A, B* $\in V^H$.

(1) *In general,* $\mu_A \leqslant \mu_B$ *iff* $A \cap \check{X} \sqsubseteq B \cap \check{X}$.
(2) *If A and B are H-fuzzy subsets of X, then*

$$\mu_A \leqslant \mu_B \quad \textit{iff} \quad A \sqsubseteq B$$

and

$$\mu_A = \mu_B \quad \textit{iff} \quad A \sim B.$$

Proof. Let $A, B \in V^H$.

(1) By Lemmas 1.4(2) and 1.10(1),

$$A \cap \check{X} \sqsubseteq B \cap \check{X}$$
$$\text{iff } \|u \in A \cap \check{X}\| \leqslant \|u \in B \cap \check{X}\| \quad \text{for all } u \in V^H$$
$$\text{iff } \|u \in A\| \wedge \|u \in \check{X}\| \leqslant \|u \in B\| \wedge \|u \in \check{X}\|$$
$$\text{for all } u \in V^H.$$

Then for every $u \in V^H$,

$$\begin{aligned}\|u \in A\| \wedge \|u \in \check{X}\| &= \bigvee_{x\in X} (\|u \in A\| \wedge \|u = \check{x}\|)\\ &= \bigvee_{x\in X} (\|\check{x} \in A\| \wedge \|u = \check{x}\|)\\ &= \bigvee_{x\in X} (\mu_A(x) \wedge \|u = \check{x}\|).\end{aligned}$$

Similarly, $\|u \in B\| \wedge \|u \in \check{X}\| = \bigvee_{x\in X} (\mu_B(x) \wedge \|u = \check{x}\|)$.

If $\mu_A \leqslant \mu_B$, then for all $u \in V^H$,

$$\begin{aligned}\|u \in A\| \wedge \|u \in \check{X}\| &= \bigvee_{x\in X} (\mu_A(x) \wedge \|u = \check{x}\|)\\ &\leqslant \bigvee_{x\in X} (\mu_B(x) \wedge \|u = \check{x}\|)\\ &= \|u \in B\| \wedge \|u \in \check{X}\|.\end{aligned}$$

Hence $A \cap \check{X} \sqsubseteq B \cap \check{X}$.

Conversely, if $A \cap \check{X} \sqsubseteq B \cap \check{X}$, then for all $x \in X$,

$$\begin{aligned}\mu_A(x) = \|\check{x} \in A\| &= \|\check{x} \in A\| \wedge \|\check{x} \in \check{X}\|\\ &\leqslant \|\check{x} \in B\| \wedge \|\check{x} \in \check{X}\|\\ &= \|\check{x} \in B\| = \mu_B(x).\end{aligned}$$

Therefore, $\mu_A \leqslant \mu_B$.

(2) Let A and B be H-fuzzy subsets of X, that is, $A \sqsubseteq \check{X}$ and $B \sqsubseteq \check{X}$. Then by Lemma 1.11(1),

$$A \sim A \cap \check{X} \quad \text{and} \quad B \sim B \cap \check{X}.$$

Since the inclusion relation $\sqsubseteq$ is transitive, $\mu_A \leqslant \mu_B$ iff $A \sqsubseteq B$, and immediately we have $\mu_A = \mu_B$ iff $A \sim B$. □

Hence there is a one-to-one correspondence between the set of all mappings from X to H and the set of all equivalent classes of H-fuzzy subsets of X with respect to the equivalence relation $\sim$. The next theorem shows that this correspondence preserves the basic set operations.

Theorem 2. *For all H-fuzzy set A, B, the following hold.*

(1) $\mu_{A\cap B} = \mu_A \wedge \mu_B$.
(2) $\mu_{A\cup B} = \mu_A \vee \mu_B$.
(3) $\mu_{\check{X}\setminus A} = \neg\, \mu_A$.

Proof. Let $A, B \in V^H$.

(1) By Lemma 1.10(1),

$$\begin{aligned}\mu_{A\cap B}(x) = \|\check{x} \in A \cap B\| &= \|\check{x} \in A\| \wedge \|\check{x} \in B\|\\ &= \mu_A(x) \wedge \mu_B(x)\end{aligned}$$

for all $x \in X$.

(2) By Lemma 1.10(2),

$$\begin{aligned}\mu_{A\cup B}(x) = \|\check{x} \in A \cup B\| &= \|\check{x} \in A\| \vee \|\check{x} \in B\|\\ &= \mu_A(x) \vee \mu_B(x)\end{aligned}$$

for all $x \in X$.

(3) By Lemma 1.10(3),

$$\begin{aligned}\mu_{\check{X}\setminus A}(x) = \|\check{x} \in \check{X}\setminus A\| &= \|\check{x} \in \check{X}\| \wedge \neg\|\check{x} \in A\|\\ &= \neg\|\check{x} \in A\| = \neg\, \mu_A(x)\end{aligned}$$

for all $x \in X$. □

4. Natural interpretation of fuzzy relations

An H-fuzzy relation is a subset of any cartesian product of sets in V^H, so it is a kind of an H-fuzzy set. In the following, let X, Y, Z be usual (crisp) sets.

Definition 4.1. We call every relation in V^H an *H-fuzzy relation*. For $R, u, v \in V^H$, R is an *H-fuzzy relation from u to v* if it is a relation from u to v in V^H, that is,

R is an H-fuzzy relation from u to v
iff $R \sqsubseteq u \times v$.

For every $R \in V^H$, the *membership function of R from X to Y* is the membership function of R on $X \times Y$, hence it is the mapping

$$\mu_R : X \times Y \to H; \quad \langle xy\rangle \mapsto \|\langle xy\rangle^{\check{}} \in R\|.$$

An *H-fuzzy relation from X to Y* is an H-fuzzy subset of $X \times Y$, so

R is an H-fuzzy relation from X to Y
iff $R \sqsubseteq (X \times Y)^{\check{}}$.

An H-fuzzy relation from u to u is called an *H-fuzzy relation on u*. Similarly, an *H-fuzzy relation on X* is an H-fuzzy relation from X to X.

We omit the superscripts of membership functions. For an H-fuzzy relation R from X to Y, μ_R always means the membership function of R on the product $X \times Y$, unless otherwise specified.

As in case of fuzzy sets, this definition is an extension of the ordinary definition of fuzzy relations. Ordinarily a fuzzy relation from X to Y is identified with its membership function from $X \times Y$ to $[0,1]$. So, an ordinary fuzzy relation from X to Y is nothing but a membership function of an H-fuzzy relation from X to Y when $H=[0,1]$. We can consider H-fuzzy relations apart from each 'underlying sets' X and Y.

Lemma 4.1. *Let R be an H-fuzzy relation and μ_R be the membership function of R from X to Y.*

(1) *R is an H-fuzzy relation from X to Y iff R is a relation from $\check{X}$ to $\check{Y}$ in V^H.*
(2) *$\mu_R\langle xy\rangle = \|\check{x}R\check{y}\|$ for all $x \in X$ and $y \in Y$.*

Proof. Let R be a relation in V^H and μ_R be its membership function on $X \times Y$.

(1) By Lemma 1.8(3), $(X \times Y)\check{} = \check{X} \times \check{Y}$. Hence R is an H-fuzzy relation from X to Y iff $R \sqsubseteq (X \times Y)\check{}$ iff $R \sqsubseteq \check{X} \times \check{Y}$ iff R is a relation from $\check{X}$ to $\check{Y}$ in V^H.
(2) By Lemma 1.8(2), we have
$$\mu_R\langle xy\rangle = \|\langle xy\rangle\check{} \in R\| = \|\langle \check{x}\check{y}\rangle^H \in R\| = \|\check{x}R\check{y}\|$$
for all $x \in X$, $y \in Y$. □

Since H-fuzzy relations are H-fuzzy sets, the basic properties of H-fuzzy sets in Section 3 also hold for H-fuzzy relations.

The *diagonal set of X* is the set $\Delta = \Delta_X = \{\langle xx\rangle; x \in X\}$. The identity relation on the check set of X is the check set of the diagonal set of X.

Lemma 4.2. *Let $\Delta = \Delta_X$ be the diagonal set of X.*

(1) $I_{\check{X}} = (\Delta_X)\check{}$.
(2) *For all $x, y \in X$, $\mu_{\check{\Delta}}(\langle xy\rangle) = \begin{cases} \mathbf{1} & (x=y) \\ \mathbf{0} & (x \neq y).\end{cases}$*

Proof. Let $\Delta = \{\langle xx\rangle; x \in X\}$.

(1) Obvious by the definitions.
(2) By Lemmas 4.1(2), (1), and 2.1(1), and Proposition 1.6,
$$\mu_{\check{\Delta}}(\langle xy\rangle) = \|xI_{\check{X}}y\| = \|\check{x} \in \check{X}\| \wedge \|\check{x} = \check{y}\| = \begin{cases} \mathbf{1} & (x=y) \\ \mathbf{0} & (x \neq y).\end{cases}$$
for all $x, y \in X$. □

Theorem 3. *If R is an H-fuzzy relation from X to Y and S is an H-fuzzy relation from Y to Z, then the composition $S \circ R$ is an H-fuzzy relation from X to Z, and*
$$\mu_{S\circ R}\langle xz\rangle = \bigvee_{y \in Y} (\mu_R\langle xy\rangle \wedge \mu_S\langle yz\rangle)$$
for all $x \in X$, $z \in Z$.

Proof. Let R, S be H-fuzzy relations from X to Y and from Y to Z, respectively.
By Lemma 4.1(1),
$$R \sqsubseteq \check{X} \times \check{Y} \quad \text{and} \quad S \sqsubseteq \check{Y} \times \check{Z}.$$
Then by Lemmas 2.2(3) and 1.8(3),
$$S \circ R \sqsubseteq \check{X} \times \check{Z} = (X \times Z)\check{}.$$
Hence $S \circ R$ is an H-fuzzy relation from X to Z.

The equation is proved as follows by Lemmas 4.1(2), 2.2(1), and 1.8(2),
$$\begin{aligned}\mu_{S\circ R}\langle xz\rangle &= \|\check{x}(S \circ R)\check{z}\| = \|\exists v(\check{x}Rv) \wedge vS\check{z})\| \\ &= \|\exists v \in \check{Y}(\langle \check{x}v\rangle \in R \wedge \langle v\check{z}\rangle \in S)\| \\ &= \bigvee_{y\in Y}(\|\langle xy\rangle\check{} \in R\| \wedge \|\langle yz\rangle\check{} \in S\|) \\ &= \bigvee_{y\in Y}(\mu_R\langle xy\rangle \wedge \mu_S\langle yz\rangle)\end{aligned}$$
for all $x \in X$, $z \in Z$. □

Hence the set of all H-fuzzy relations between usual sets are closed under composition, and the equation in the theorem coincides with the defining equation of the so-called max–min composition for ordinary fuzzy relations.

Theorem 4. *If R is an H-fuzzy relation from X to Y, then the inverse relation R^{-1} in V^H is an H-fuzzy relation from Y to X, and for all $x \in X, y \in Y$,*
$$\mu_{R^{-1}}\langle yx\rangle = \mu_R\langle xy\rangle.$$

Proof. Let R be an H-fuzzy relation from X to Y. Then by Lemma 4.1(1), $R \sqsubseteq \check{X} \times \check{Y}$. By Lemmas 2.4(3) and 1.8(3),

$$R^{-1} \sqsubseteq \check{Y} \times \check{X} = (Y \times X)\check{}.$$

Hence R^{-1} is an H-fuzzy relation from Y to X by Lemma 2.4(2).

As for the equation, by Lemma 4.1(2) and Lemma 2.4(1),

$$\mu_{R^{-1}}\langle yx\rangle = \|\check{y}R^{-1}\check{x}\| = \|\check{x}R\check{y}\| = \mu_R\langle xy\rangle$$

for all $x \in X$, $y \in Y$. □

Hence the set of all H-fuzzy relations between usual sets are also closed under inverse, and the equation in the theorem coincides with the defining equation of inverse in most of the ordinary fuzzy literature (in [3] it is called converse). Therefore, we can define composition and inverse for H-fuzzy relations, which seem to be most natural as extensions of those for ordinary fuzzy relations.

Now an H-fuzzy relation on a usual set is called *reflexive* when it is reflexive as a relation in V^H. Similarly, it is defined to be *symmetric*, *transitive*, *antisymmetric*, and *connected* if it is so in V^H. Hence an H-fuzzy relation on a usual set is an *H-fuzzy equivalence relation* if it is reflexive, symmetric, and transitive, and is an *H-fuzzy order relation* if it is reflexive, antisymmetric, and transitive.

Theorem 5. *Let R be an H-fuzzy relation on X. Then the following hold.*

(1) *R is reflexive iff* $\mu_R\langle xx\rangle = \mathbf{1}$ *for all* $x \in X$.
(2) *R is symmetric iff* $\mu_R\langle xy\rangle = \mu_R\langle yx\rangle$ *for all* $x, y \in X$.
(3) *R is transitive*
iff $\mu_R\langle xy\rangle \wedge \mu_R\langle yz\rangle \leqslant \mu_R\langle xz\rangle$ *for all* $x, y, z \in X$
iff $\bigvee_{y \in Y} (\mu_R\langle xy\rangle \wedge \mu_R\langle yz\rangle) \leqslant \mu_R\langle xz\rangle$ *for all* $x, z \in X$.
(4) *R is antisymmetric iff* $x \neq y$
implies $\mu_R\langle xy\rangle \wedge \mu_R\langle yx\rangle = \mathbf{0}$ *for all* $x, y \in X$.
(5) *R is connected iff* $\mu_R\langle xy\rangle \vee \mu_R\langle yx\rangle = \mathbf{1}$ *for all* $x, y \in X$.

Proof. Let R be an H-fuzzy relation on X. Then since $R \sqsubseteq \check{X} \times \check{X}$,

$$\|uRv\| = \|uRv \wedge u \in \check{X} \wedge v \in \check{X}\|$$

$$= \bigvee_{x, y \in X} (\|\check{x}R\check{y}\| \wedge \|u = \check{x}\| \wedge \|v = \check{y}\|)$$

for all $u, v \in V^H$. We omit the proof of (1) and (2), for the proofs are easy and similar to the other cases.
(3) By Lemmas 2.6(3) and 4.1(2),

R is transitive
iff $\|uRv\| \wedge \|vRw\| \leqslant \|uRw\|$ for all $u, v, w \in V^H$
iff $\|\check{x}R\check{y}\| \wedge \|\check{y}R\check{z}\| \leqslant \|\check{x}R\check{z}\|$ for all $x, y, z \in X$
iff $\mu_R\langle xy\rangle \wedge \mu_R\langle yz\rangle \leqslant \mu_R\langle xz\rangle$ for all $x, y, z \in X$
iff $\bigvee_{y \in Y} (\mu_R\langle xy\rangle \wedge \mu_R\langle yz\rangle) \leqslant \mu_R\langle xz\rangle$
for all $x, z \in X$.

(4) By Lemmas 2.6(4), and 4.1(2), and Proposition 1.6,

R is antisymmetric
iff $\|uRv\| \wedge \|vRu\| \leqslant \|u = v\|$ for all $u, v \in V^H$
iff $\|\check{x}R\check{y}\| \wedge \|\check{y}R\check{x}\| \leqslant \|\check{x} = \check{y}\|$ for all $x, y \in X$
iff $x \neq y$ implies $\mu_R\langle xy\rangle \wedge \mu_R\langle yx\rangle = \mathbf{0}$
for all $x, y \in X$.

(5) By Lemmas 2.6(5), and 4.1(2), and Proposition 1.6

R is connected
iff $\|u \in \check{X}\| \wedge \|v \in \check{Y}\| \leqslant \|uRv\| \vee \|vRu\|$
for all $u, v \in V^H$
iff $\|\check{x} \in \check{X}\| \wedge \|\check{y} \in \check{Y}\| \leqslant \|\check{x}R\check{y}\| \vee \|\check{y}R\check{x}\|$
for all $x, y \in X$
iff $\|\check{x}R\check{y}\| \vee \|\check{y}R\check{x}\| = \mathbf{1}$ for all $x, y \in X$
iff $\mu_R\langle xy\rangle \vee \mu_R\langle yx\rangle = \mathbf{1}$ for all $x, y \in X$. □

The first four conditions essentially coincide with the usual defining conditions of corresponding notions of ordinary fuzzy relation [1,7,8,24,28]. Strictly speaking, the condition for antisymmetric is in different form, for in ordinary fuzzy literature, R is defined to be antisymmetric iff $\mu_R\langle xy\rangle > 0$ and $\mu_R\langle yx\rangle > 0$ implies $x = y$ for all $x, y \in X$. But it is easy to verify that this is equivalent to the condition in (4) in case $H = [0, 1]$.

On the other hand, the condition in (5) is not equivalent to the ordinary definition even if $H=[0,1]$. In ordinary fuzzy literature such as [1,24,38], (though the word 'connected' is not used,) the additional condition for linear order is written as: $x \neq y$ implies $\mu_R\langle xy\rangle > 0$ or $\mu_R\langle yx\rangle > 0$ for all $x, y \in X$. This is properly weaker than the condition in (5) when $H=[0,1]$. Therefore, the usual notion of linear order in ordinary fuzzy relation is weaker than that in our model.

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