Last name ______________________

First name ______________________

LARSON—MATH 656—CLASSROOM WORKSHEET 21
Gallai-Edmonds-Decomposition.

Organizational Notes

1. Don't forget to send your Notes / Classroom worksheet after each class (make the email subject useful: like "Math 656 c21 notes").
2. The VCU Discrete Math Seminar is every Wednesday.
3. $h07$ (the Gallai-Edmonds decomposition worksheet) is due on Wednesday.
4. Read ahead! Next up we'll talk about Petersen's Theorem (Corollary to Tutte's Theorem in Sec 3.3) and then Network Flow problems (Sec. 4.3)

Concepts & Notation

- Edmonds-Gallai Decomposition (West paper).
- Petersen's Theorem (Sec. 3.3).
- Network Flows (Sec. 4.3).

Review

1. A vertex v in a graph is either (1) covered by every maximum matching (set B), or (2) not covered by every maximum matching (set D). A vertex in B either (1) has a neighbor outside B (set A) or (2) does not (set C). The **Gallai-Edmonds Decomposition** is the partition of $V(G)$ into sets C, A and D.
2. One (efficient) algorithm for finding the Gallai-Edmonds Decomposition is simple to test each vertex v to see whether it is in D. Then A must be the vertices adjacent to the vertices in D and C must be the remaining vertices ($C = V - A - D$).
3. (**Gallai-Edmonds Structure Theorem**). Let A, C, D, be the sets in the Gallai-Edmonds Decomposition of a graph G. Let $G_1,\ldots,G_k$ be the components of $G[D]$. If M is a maximum matching in G then:
   (a) M covers C and matches A into distinct components of $G[D]$.
   (b) Each G_i is factor-critical and M restricts to a near-perfect matching on G_i,
   (c) If $S \subseteq A$ is non-empty then $N_G(S)$ has a vertex in at least $|S|+1$ of $G_1,\ldots,G_k$,
   (d) $def(A) = def(G) = k - |A|$.

Notes

West's proof

1. Let M be a maximum matching M of a graph G with decomposition sets, C, A, D

2. Define T as in the proof of the Berge-Tutte formula proof (we'll also need facts about the auxiliary graph $H(T)$),

3. We also know:

   (a) All components of $G - T$ are factor-critical (and hence odd),

   (b) Any maximum matching matches T to one vertex in each of T components of $G - T$ (in particular M).

4. Define $R \subseteq T$ to be a maximal subset with $|N_{H(T)}(R)| = |R|$.

5. Let R' be the union of the components corresponding to the vertices R matches in $H(T)$ with respect to M.

6. Argue that $R \cup R' \subseteq C$ (and later $R \cup R' = C$).

7. Let $D' = V(G) - T - R'$ and argue $D = D'$.

8. Argue $A = T - R$.

9. (**Gallai-Edmonds Structure Theorem**). Let A, C, D, be the sets in the Gallai-Edmonds Decomposition of a graph G. Let $G_1,\ldots,G_k$ be the components of $G[D]$. If M is a maximum matching in G then:

   (a) M covers C and matches A into distinct components of $G[D]$.

   (b) Each G_i is factor-critical and M restricts to a near-perfect matching on G_i,

   (c) If $S \subseteq A$ is non-empty then $N_G(S)$ has a vertex in at least $|S|+1$ of $G_1,\ldots,G_k$,

   (d) $def(A) = def(G) = k - |A|$.

10. (**Petersen's Theorem**) If a graph has a perfect matching and no cut edges then it has a perfect matching.