CHAPTER 45

Working with Calculus

This final chapter emphasizes two things. First, our present collection of integration formulas and techniques are remarkably limited; section 45.1 introduces a few additional minor methods, but much more will come in Calculus II. Second, the limit definitions of the derivative and definite integral are actually quite significant, and deserve to be remembered and internalized. This point is illustrated in Section 45.3, which applies calculus to the problem of finding the average value of a function.

45.1 What to Do when Substitution Fails

In any textbook, the exercises in the section on substitution can all be done with substitution. This can lend the erroneous impression that substitution is the ultimate integration technique. In reality, substitution has severe limitations. For it to work, the integral must have form $\int f\big(g(x)\big)\, g'(x)\, dx$ (or can be put into this form), so that a substitution $u = g(x)$ converts it to $\int f(u)\, du$. **In addition** we must be able to do $\int f(u)\, du$. If at least one of these conditions is not met, substitution will not work. Here is a sampling of just a few integrals for which substitution does not apply:

$$\int \cos(x^2)x^2\, dx \qquad \int \cos^2(x)\, dx \qquad \int \ln(x)\, dx \qquad \int \sec(x)\, dx$$

Integrals such as these will have to wait until Calculus II. (Though technically, you can write a formula for each of these now. For example, the fundamental theorem of calculus (part 1) yields

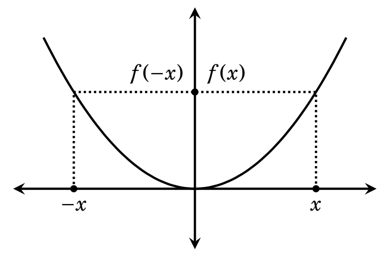
$$\int \cos(x^2)x^2\, dx \;=\; \int_0^x \cos(t^2)t^2\, dt + C,$$

so the integral $\int \cos(x^2)x^2\, dx$ *does* exist. But we'd like an answer that doesn't have an integral in it. That is what has to wait for Calculus II.)

But there are several specialized techniques that can be applied in conjunction with our present setting of Calculus I. We discuss them now.

Sometimes certain properties of a function can assist in evaluating certain integrals of it. Such is the case for *even* and *odd* functions.

Recall that a function f is **even** if $f(-x) = f(x)$ for all x in its domain. A typical even function is shown on the right. Any two opposite input values x and $-x$ have equal output $f(-x) = f(x)$, so even functions are symmetric about the y-axis.

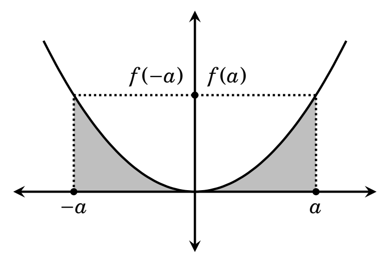


For example, $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$. Also $\cos(x)$ is even because of the identity $\cos(-x) = \cos(x)$. Other even functions include $f(x) = x^4 + x^2 + 1$ and $g(x) = x^2 + \cos(x)$.

Symmetry yields the following fact about even functions.

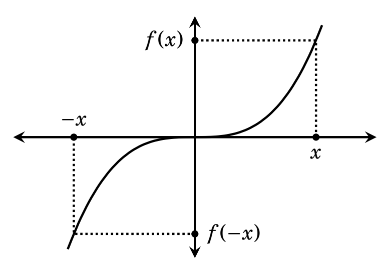
Fact 45.1 If f is an even function, continuous on $[-a,a]$, then

$$\int_{-a}^{a} f(x)\,dx \;=\; 2\int_{0}^{a} f(x)\,dx$$



Though sometimes helpful, this is not the most useful fact. After all, if are trying to find $\int_{-a}^{a} f(x)\,dx$, then $2\int_0^a f(x)\,dx$ is not likely to be any easier. But the corresponding fact for odd functions gives a definitive answer.

A function is **odd** if $f(-x) = -f(x)$ for all x in its domain. A typical odd function is shown on the right. Any two opposite input values x and $-x$ have opposite outputs $f(x)$ and $-f(x)$, so odd functions are symmetric with respect to the origin.

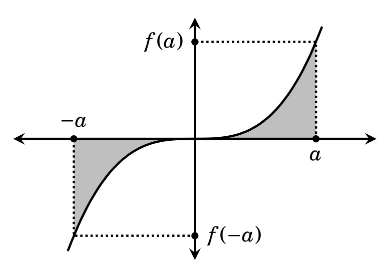


Examples of odd functions are $f(x) = x^3$, $f(x) = x^5 + x^3 + 4x$ and $f(x) = \sin(x)$.

Symmetry yields the following fact about odd functions.

Fact 45.2 If f is an odd function, continuous on $[-a,a]$, then

$$\int_{-a}^{a} f(x)\,dx \;=\; 0$$



Example 45.1 Find $\displaystyle\int_{-\pi}^{\pi} x^3 \sin\left(x^2\right)\,dx$.

Solution The integrand $x^3 \sin\left(x^2\right)$ is not one whose antiderivativea we can compute with any of our integral formulas. Therefore the fundamental theorem of calculus is of no use here.

Notice however, that the integrand $f(x) = x^3 \sin\left(x^2\right)$ is *odd* because

$$f(-x) = (-x)^3 \sin\left((-x)^2\right) = -x^3 \sin\left(x^2\right) = -f(x).$$

Therefore Fact 45.2 gives an answer of $\displaystyle\int_{-\pi}^{\pi} x^3 \sin\left(x^2\right)\,dx = 0.$ ✍

In order for Fact 45.2 to work, the integrand must be odd **and** the two limits of integration must be negatives of each other. We are can't compute, for example, $\int_{-\pi}^{2\pi} x^3 \sin\left(x^2\right)\,dx$. So Fact 45.2 is of limited utility. It is a little-used tool that is nonetheless perfect for just the right job.

Another minor tool in our integration toolbox is the use of trig identities. To illustrate this, consider the integrals $\int \sin^2(x)\,dx$ and $\int \cos^2(x)\,dx$. We don't have integration formulas for these, and substitution doesn't apply. The issue is the powers of 2; if they were not there the integrals would be easy. But these powers of 2 can be eliminated with the following trig identities

$$\boxed{\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}}$$

(These are identities (3.17) and (3.16) from page 46.)

Example 45.2 Find $\displaystyle\int \sin^2(x)\,dx$.

Solution
$$\begin{aligned}\int \sin^2(x)\,dx &= \int \left(\frac{1-\cos(2x)}{2}\right) dx = \frac{1}{2}\int \left(1-\cos(2x)\right) dx \\ \frac{1}{2}\left(x + \frac{1}{2}\cos(2x)\right) + C &= \boxed{\frac{1}{2}x + \frac{1}{4}\cos(2x) + C.}\end{aligned}$$ ✍

To repeat, the methods (odd functions and identities) in this section are minor. For the purposes of Calculus I, your attention should be focused mostly on the standard integration formulas, including substitution. To underscore this, the integration exercises for this chapter are a cumulative mix. For each one, decide which technique applies and execute it. This kind of critical analysis is an important test-taking skill. (More broadly, it is a crucial life skill!)

45.2 Why Definitions Are Important

The two major concepts in calculus were defined by limits. First, the derivative of a function f was defined as

$$f'(x) \;=\; \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

But once we developed our derivative rules we rarely had occasion to use the limit again. Likewise the definite integral of f, over $[a,b]$ is

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x,$$

though we almost never evaluated this limit directly. There was no need to do so once we had the fundamental theorem of calculus.

You can legitimately ask why the limits are necessary. One reason is that the limits give meaning to derivatives and integrals. The limit for $f'(x)$ is the formula for the slope of the tangent to $y=f(x)$ at $\big(x,f(x)\big)$, so the limit gives the derivative its slope interpretation. And the limit for $\int_a^b f(x)dx$ is the formula for area, so it gives the integral its area interpretations.

But here is a more significant reason that the limits are important: In modeling a real-world problem you may find that the solution takes the form of one of the above limits. When this happens, calculus can be applied to the problem.

This happened in Chapter 26 when we examined motion on a line. We sought a formula for velocity, and our model led to $v(t)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$, where f is the object's position. Once this limit formula for the derivative came into the picture, we saw that velocity is $f'(t)$, and we could then use calculus (derivative rules, etc.) in motion problems.

Likewise, if you continue studying (and using) calculus and its applications, you will encounter situations in which the model of a problem leads to a limit of the form $\lim\limits_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$. When this happens, you'll know that this is a definite integral, and calculus can be applied.

So the limits are a vital bridge between applications and calculus.

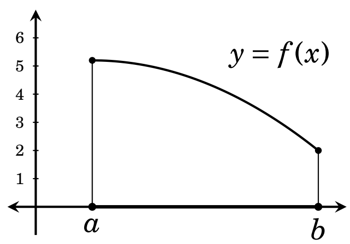
We will conclude with just one more instance of this, though you will see further instances in Calculus II. Our final section takes up the problem of finding the average value of a function on an interval. In modeling the problem, we will discover that the answer involves the limit $\lim\limits_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$, which is then replaced by $\int_a^b f(x)dx$. At this point the problem is squarely within the domain and scope of calculus.

45.3 Average Value of a Function

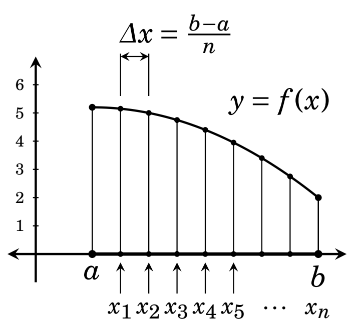
Consider the problem of finding the average temperature on a certain day. You would add up all the temperatures at each instant of the day, and then divide by the number of instants in the day. There are infinitely many instants in a day, so this quotient would $\frac{\infty}{\infty}$.

That may sound paradoxical, but integration makes sense of it. This section explains how.

In general we are concerned with finding the average value of a function $f(x)$ on an interval $[a,b]$, like the one shown below.



To find the average value of $f(x)$ on $[a,b]$ we could begin by taking n sample x-values $x_1, x_2, x_3, \ldots, x_n$ in the interval. To ensure an unbiased sample, make them evenly spaced a distance of $\Delta x = \frac{b-a}{n}$ apart.

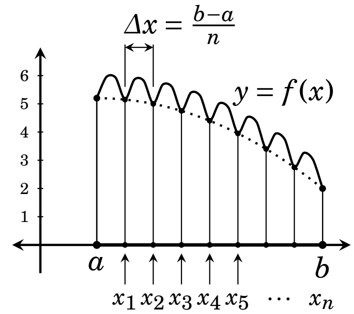


Then the average value of $f(x)$ on $[a,b]$ is approximately

$$\begin{aligned}\frac{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)}{n} &= \sum_{k=1}^{n} f(x_k)\frac{1}{n} \\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\frac{b-a}{n} \;=\; \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x.\end{aligned}$$

So the average value of $f(x)$ on $[a,b]$ is approximately $\dfrac{1}{b-a}\displaystyle\sum_{k=1}^{n} f(x_k)\Delta x$. Notice the Riemann sum here. A definite integral is beginning to emerge.

But $A = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x$ is only approximates of the average value of $f(x)$ on $[a,b]$. For all we know, our sample points may have hit the function at low points, as shown below. This would skew A away from the true average.



The obvious way to overcome this problem is to increase the number of sample points. To get the average value exactly, let the number n of sample points approach infinity. Then the average value is exactly

$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x \;=\; \frac{1}{b-a}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x\right) \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This is our formula for average value.

Fact 45.3 Average Value of a Function

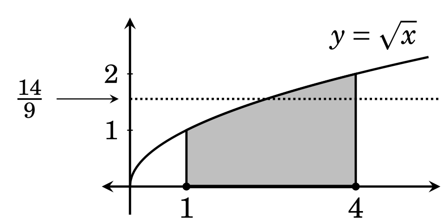
The average value of $f(x)$ on the interval $[a,b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx.$

(We assume f is continuous on $[a,b]$, so that the integral exists.)

Example 45.3 Find the average value of $\sqrt{x}$ on the interval $[1,4]$.

Solution The average is $\dfrac{1}{4-1}\displaystyle\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\left[\frac{2}{3}\sqrt{x}^3\right]_1^4 = \frac{1}{3}\left(\frac{2}{3}\sqrt{4}^3 - \frac{2}{3}\sqrt{1}^3\right) = \frac{14}{9}.$

The function and interval are shown on the right. Notice that our answer of $14/9 = 1.\overline{5}$ for the average value of $\sqrt{x}$ on $[1,5]$ is entirely reasonable. The values of $\sqrt{x}$ above and below $14/9$ appear to balance out. ✍



Exercises for Chapter 45

In exercises 1–12, use any applicable technique to find the definite integral. (These problems are cumulative. A variety of techniques may apply.)

1. $\int_{-\pi}^{\pi}\left(x+x^7\cos(x)\right)\,dx$

2. $\int_{-1}^{1}\left(e^{\cos(x)}\sqrt[3]{x}\right)\,dx$

3. $\int_{-1}^{1}(2x-1)e^{x^2-x}\,dx$

4. $\int_{-\pi/4}^{\pi/4}\tan\left(x^3\right)\,dx$

5. $\int_{-1}^{1}4x^3\,e^{x^4-x^2}\,dx$

6. $\int_{-\pi/4}^{\pi/4}x\tan\left(x^2\right)\,dx$

7. $\displaystyle\int_{-1}^{1}\frac{x^2+3}{x^3+x}\,dx$

8. $\displaystyle\int_{-1}^{1}\frac{t}{\sqrt{t^4+1}}\,dt$

9. $\int_{-\pi}^{\pi}\cos^2(x)\,dx$

10. $\int_{-\pi}^{\pi}\cos^2(5x)\,dx$

11. $\int_{-5}^{5}\sin\left(x^3+x\right)\,dx$

12. $\int_{-\pi}^{\pi}\sin^2(x)\,dx$

In exercises 13–22, find the average value of the function on the given interval.

13. $\sin(x)$ on $[0,\pi]$ and on $[0,2\pi]$

14. $1/x$ on $[1,5]$

15. $\sin\left(x^7\right)$ on $[-2\pi,2\pi]$ is

16. $x\sin\left(x^8\right)$ on $[-1,1]$

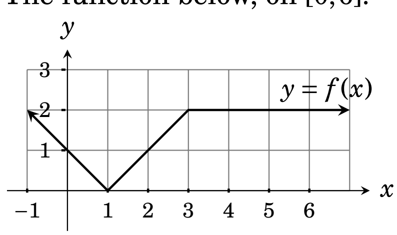
17. $\sec(x)\tan(x)$ on $[-\pi/4,0]$

18. $\sec^2(x)$ on $[-\pi/4,0]$

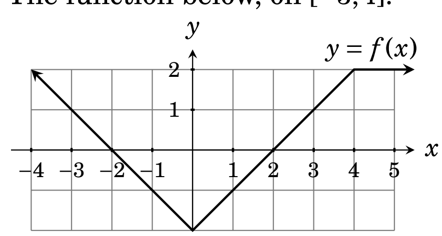
19. x^2 on $[0,9]$

20. $\frac{1}{1+x^2}$ on $[0,1]$

21. The function below, on $[0,6]$.



22. The function below, on $[-3,4]$.



23. Find $\int_{-5}^{5}\left(3x^5+x\right)\,dx$ two ways: First, use FTC Part 2. Then use Fact 45.2.

24. Find the equation of the tangent line to the graph of $f(x)=\displaystyle\int_{-2}^{x}\frac{t^3}{\sqrt{t^2+5}}\,dt$ at the point $(2,f(2))$.

25. A kiln has a temperature of $70+3t^2$ degrees F. at time t. Find the average temperature of the kiln between times 0 and 2.

Exercise Solutions for Chapter 45

1. The integrand $f(x) = x+x^7\cos(x)$ is odd, as $f(-x) = -x+(-x)^7\cos(-x) = -x-x^7\cos(x) = -\left(x+x^7\cos(x)\right) = -f(x)$. Therefore $\int_{-\pi}^{\pi}\left(x+x^7\cos(x)\right)\,dx = 0$ by Fact 45.2.

3. $\int_{-1}^{1}(2x-1)e^{x^2-x}\,dx$ (The integrand is not odd.) Let $u = x^2-x$, so $du = (2x-1)dx$.
$\int_{-1}^{1}(2x-1)e^{x^2-x}\,dx = \int_{-1}^{1}e^{x^2-x}(2x-1)\,dx = \int_{(-1)^2-(-1)}^{1^2-1}e^u\,du = \int_{1}^{0}e^u\,du = e^0-e^1 = 1-e$

5. $\int_{-1}^{1}4x^3e^{x^4-x^2}\,dx$ Subsituting $u = x^4-x^2$ does not give a match for du, but the integrand $f(x) = 4x^3e^{x^4-x^2}$ is odd: $f(-x) = 4(-x)^3e^{(-x)^4-(-x)^2} = -4x^3e^{x^4-x^2} = -f(x)$. Therefore $\int_{-1}^{1}4x^3e^{x^4-x^2}\,dx = 0$.

7. $\displaystyle\int_{-1}^{1}\frac{x^2+3}{x^3+x}\,dx$ The integrand $f(x) = \dfrac{x^2+3}{x^3+x}$ is odd, as $f(-x) = \dfrac{(-x)^2+3}{(-x)^3+(-x)} = \dfrac{x^2+3}{-x^3-x}$ $= -\dfrac{x^2+3}{x^3+x} = -f(x)$. Therefore $\displaystyle\int_{-1}^{1}\frac{x^2+3}{x^3+x}\,dx = 0$

9. $\int_{-\pi}^{\pi}\cos^2(x)\,dx$ The integrand is not odd, and substitution fails. We will use the identity $\cos^2(x) = \left(1+\cos(2x)\right)/2$.
$\displaystyle\int_{-\pi}^{\pi}\cos^2(x)\,dx = \int_{-\pi}^{\pi}\frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\int_{-\pi}^{\pi}1+\cos(2x)\,dx = \frac{1}{2}\left[x+\frac{1}{2}\sin(2x)\right]_{-\pi}^{\pi}dx =$
$\displaystyle\frac{1}{2}\left(\left(\pi+\frac{1}{2}\sin(2\pi)\right)-\left(-\pi+\frac{1}{2}\sin(-2\pi)\right)\right) = \frac{1}{2}\Big((\pi+0)-(-\pi+0)\Big) = \pi.$

11. $\int_{-5}^{5}\sin\left(x^3+x\right)\,dx$ Notice $f(x) = \sin\left(x^3+x\right)$ is odd, as $f(-x) = \sin\left((-x)^3+(-x)\right) = \sin(-(x^3+x)) = -\sin\left(x^3+x\right) = -f(x)$. Thus $\int_{-5}^{5}\sin\left(x^3+x\right)\,dx = 0$.

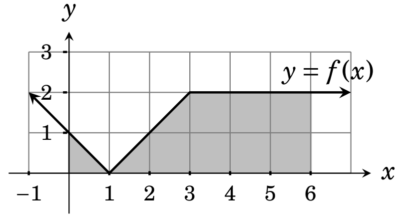
13. Average value on $[0,\pi]$ is $\frac{1}{\pi-0}\int_0^{\pi}\sin(x)\,dx = \frac{1}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{1}{\pi}\Big(-\cos(\pi)-\big(-\cos(0)\big)\Big) = \frac{2}{\pi}$.
Average value on $[0,2\pi]$ is $\frac{1}{2\pi-0}\int_0^{2\pi}\sin(x)dx = \frac{1}{2\pi}\Big[-\cos(x)\Big]_0^{2\pi} = \frac{1}{2\pi}\Big(-\cos(2\pi)-\big(-\cos(0)\big)\Big) = 0$.

15. Average value of $\sin\left(x^7\right)$ on $[-2\pi,2\pi]$ is $\displaystyle\frac{1}{\pi-(-\pi)}\int_{-2\pi}^{2\pi}\sin\left(x^7\right)\,dx = \frac{1}{2\pi}\cdot 0 = 0.$
(The integrand is odd.)

17. The average value of $\sec(x)\tan(x)$ on $[-\pi/4,0]$ is $\displaystyle\frac{1}{0-(-\pi/4)}\int_{-\pi/4}^{0}\sec(x)\tan(x)\,dx =$
$\displaystyle\frac{4}{\pi}\Big[\sec(x)\Big]_{-\pi/4}^{0} = \frac{4}{\pi}\big(\sec(0)-\sec(-\pi/4)\big) = \frac{4}{\pi}\left(1-\sqrt{2}\right).$

19. Average value of x^2 on $[0,9]$ is $\displaystyle\frac{1}{9-0}\int_0^9 x^2\,dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_0^9 = 27.$

21. Find the average value of the function graphed below, on $[0,6]$.



Answer: $\displaystyle\frac{1}{6-0}\int_0^6 f(x)\,dx = \frac{1}{6}\cdot\frac{17}{2} = \frac{17}{12}$

23. Find $\int_{-5}^{5}\left(3x^5+x\right)\,dx$ two ways.

By FTC Part 2, $\int_{-5}^{5}\left(3x^5+x\right)\,dx = \left[\frac{x^6}{2}+\frac{x^2}{2}\right]_{-5}^{5} = \left(\frac{5^6}{2}+\frac{5^2}{2}\right)-\left(\frac{(-5)^6}{2}+\frac{(-5)^2}{2}\right)=0.$

Alternatively, notice that the integrand $f(x)=3x^5+x$ is odd because $f(-x)=3(-x)^5-x=-3x^5-x=-\left(3x^5+x\right)=-f(x)$. Then by Fact 45.2, $\int_{-5}^{5}\left(3x^5+x\right)\,dx=0$.

25. A kiln has a temperature of $70+3t^2$ degrees F. at time t. Find the average temperature of the kiln between times 0 and 2.

$$\frac{1}{2-0}\int_0^2\left(70+3t^2\right)\,dx=\frac{1}{2}\left[70t+t^3\right]_0^2=\frac{1}{2}\left(70\cdot 2+2^3\right)=74°\text{F}.$$