CHAPTER 23

The Chain Rule

We have so far developed 14 rules for finding derivatives without limits. With them we can differentiate a great many functions. For instance, imagine we need to differentiate $\sin(x)\left(x^2+x\right)$. We recognize this as a product of the functions $\sin(x)$ and x^2+x, and thus use the product rule:

$$\frac{d}{dx}\Big[\sin(x)\left(x^2+x\right)\Big] = \cos(x)\left(x^2+x\right)+\sin(x)(2x+1).$$

But what if instead we needed $\frac{d}{dx}\Big[\sin\left(x^2+x\right)\Big]$? Notice that $\sin\left(x^2+x\right)$ is not a *product* $f(x)g(x)$, but rather a *composition* $f(g(x))$:

$$\begin{array}{l} \dfrac{d}{dx}\Big[\sin\big(x^2+x\big)\Big] \\ \qquad\ \uparrow \qquad \uparrow \\ \dfrac{d}{dx}\Big[\ f\ \big(\ g(x)\ \big)\Big] \end{array}$$

We can't do this because we don't have a rule for $\frac{d}{dx}\left[f\left(g(x)\right]\right.$. That's our goal.

Chapter goal: Develop a rule for $\dfrac{d}{dx}\Big[f\big(g(x)\big)\Big]$.

This new "composition rule" will be called the **chain rule**. Let's get straight to our task and derive it. We will use Definition 16.1:

$$f'(x) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}.$$

But let's emphasize the structure of this by replacing z and x with $\square$ and $\blacksquare$:

$$f'(\blacksquare) = \lim_{\square\to\blacksquare}\frac{f(\square)-f(\blacksquare)}{\square-\blacksquare}. \qquad (*)$$

In words, "*If* $\square$ *approaches* $\blacksquare$, *then* $\frac{f(\square)-f(\blacksquare)}{\square-\blacksquare}$ *approaches* $f'(\blacksquare)$."

The first step in finding $\frac{d}{dx}\left[f\big(g(x)\big]\right.$ is to apply the definition of a derivative (Definition 16.1) to the function $f\big(g(x)\big)$.

$$\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = \lim_{z\to x}\frac{f\big(g(z)\big)-f\big(g(x)\big)}{z-x}$$

Now multiply this by $1=\frac{f(z)-f(x)}{f(z)-f(x)}$, then rearrange and apply a limit law:

$$\begin{aligned} &= \lim_{z\to x}\frac{f\big(g(z)\big)-f\big(g(x)\big)}{g(z)-g(x)}\cdot\frac{g(z)-g(x)}{z-x} \\ &= \lim_{z\to x}\frac{f\big(g(z)\big)-f\big(g(x)\big)}{g(z)-g(x)}\cdot\lim_{z\to x}\frac{g(z)-g(x)}{z-x}\end{aligned}$$

Notice that the limit on the right equals $g'(x)$, so let's make that replacement:

$$= \lim_{z\to x}\frac{f\big(g(z)\big)-f\big(g(x)\big)}{g(z)-g(x)}\cdot g'(x)$$

To highlight the remaining limit's structure, put the two occurrences of $g(z)$ in white boxes and the two occurrences of $g(x)$ in gray boxes:

$$= \lim_{z\to x}\frac{f\left(\boxed{g(z)}\right)-f\left(\boxed{g(x)}\right)}{\boxed{g(z)}-\boxed{g(x)}}\cdot g'(x)$$

In the limit, z approaches x, and as this happens $\boxed{g(z)}$ approaches $\boxed{g(x)}$. By (*) on the previous page, the limit equals $f'\left(\boxed{g(x)}\right)$. The above becomes

$$= f'\left(\boxed{g(x)}\right)\cdot g'(x).$$

We have just shown $\frac{d}{dx}\left[f\big(g(x)\big]\right. = f'\big(g(x)\big)\cdot g'(x)$. This is the **chain rule**.[1] Actually, we will have two versions of the chain rule. This is Version 1.

Chain Rule (Verison 1): $\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big)\cdot g'(x)$

[1]Our derivation of the chain rule was quick and informal, and not particularly robust. There are some caveats. For one, if g is a constant function (or constant near x), then we have $g(z)-g(x)=0$ on the denominator and the whole calculation comes crashing down. This can be fixed in a more careful proof. Notice also that we are quietly assuming that $g'(x)$ exists, so g is continuous at x; Therefore $z\to x$ really does force $g(z)\to g(x)$, as asserted.

For example, let's apply this pattern to find the derivative of $\sin(x^2+x)$. (The problem from this chapter's first paragraph.) The chain rule gives the answer in one step:

$$\begin{aligned}
\frac{d}{dx}\Big[\sin\big(x^2+x\big)\Big] &= \cos\big(x^2+x\big)\,(2x+1)\\
\frac{d}{dx}\Big[\,f\,\big(g(x)\big)\Big] &= f'\big(g(x)\big)\,g'(x)
\end{aligned}$$

Here are two more examples of this simple pattern $\frac{d}{dx}[f(g(x)] = f'(g(x))\cdot g'(x)$.

$$\frac{d}{dx}\Big[\tan\big(x^5\big)\Big] = \sec^2\big(x^5\big)\,5x^4$$

$$\frac{d}{dx}\Big[\cos\big(e^x\big)\Big] = -\sin\big(e^x\big)\,e^x$$

Before doing more examples, we will discuss Version 2 of the chain rule. It says exactly the same thing as Version 1, but in a different way.

Split the composition $y = f(g(x))$ into two equations, as $\begin{cases} y = f(u) \\ u = g(x). \end{cases}$ These two functions have derivatives $\frac{dy}{du} = f'(u)$ and $\frac{du}{dx} = g'(x)$. The chain rule says

$$\begin{aligned}
\frac{d}{dx}\Big[f\big(g(x)\big)\Big] &= f'\big(g(x)\big)\cdot g'(x), && \text{(Version 1)}\\
\frac{d}{dx}\Big[f\big(g(x)\big)\Big] &= f'\big(u\big)\cdot g'(x), && \text{(because } u = g(x))\\
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && \text{(changing notation)}
\end{aligned}$$

In summary, if $y = f(g(x))$, then $y = f(u)$, where $u = g(x)$, and $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$. This is Version 2 of the chain rule. Let's now formally state both versions.

> **Rule 15** (The Chain Rule)
>
> **Version 1**: $\dfrac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big)\cdot g'(x)$
>
> **Version 2**: If $y = f(g(x))$, then $\begin{cases} y = f(u) \\ u = g(x) \end{cases}$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$.

As you do exercises, you will find that sometimes one version is more convenient than the other, depending on the problem. Our first few examples will solve each problem both ways.

Example 23.1 Find the derivative of $y = \sin(x^2 + x)$.

This is a composition $y = f\big(g(x)\big)$, so we need to use the chain rule. We can get the answer immediately with Version 1:

$$\frac{d}{dx}\Big[\sin\big(x^2+x\big)\Big] = f'\big(g(x)\big)g'(x) = \boxed{\cos\big(x^2+x\big)(2x+1).}$$

Now we'll do exactly the same problem, but with Version 2 of the chain rule. We need to find the derivative of $y = \sin\big(x^2+x\big)$. Write this as $\begin{cases} y = \sin(u) \\ u = x^2+x. \end{cases}$ These two individual functions have derivatives $\frac{dy}{du} = \cos(u)$ and $\frac{du}{dx} = 2x+1$. Version 2 of the chain rule says

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= \cos(u)\cdot(2x+1). \end{aligned}$$

This expression has two variables, u and x. But remember that $u = x^2 + 1$. Making that substitution yields the answer:

$$= \boxed{\cos\big(x^2+1\big)\cdot(2x+1).}$$

✍

Example 23.2 Find the derivative of $y = \sec\big(x^3 + e^x\big)$.

This is a composition, so the chain rule is the right tool for this problem. We will solve this problem two ways, using both versions of the chain rule.

Let's do Version 2 first. Write $y = \sec\big(x^3 + e^x\big)$ as $\begin{cases} y = \sec(u) \\ u = x^3 + e^x. \end{cases}$ These two functions have derivatives $\frac{dy}{du} = \sec(u)\tan(u)$ and $\frac{du}{dx} = 3x^2+e^x$. Version 2 says

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= \sec(u)\tan(u)\cdot\big(3x^2+e^x\big) \\ &= \boxed{\sec\big(x^3+e^x\big)\tan\big(x^3+e^x\big)\big(3x^2+e^x\big)} \qquad \text{(plug in } u = x^3+e^x\text{)} \end{aligned}$$

Now let's use Version 1, which says $\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big)\cdot g'(x)$. But in this problem $f'(x) = \sec(x)\tan(x)$, so are *two* places to plug in the $g(x)$. We get

$$\frac{d}{dx}\Big[\sec\big(x^3+e^x\big)\Big] = \boxed{\sec\big(x^3+e^x\big)\tan\big(x^3+e^x\big)(3x^2+e^x).}$$

✍

Example 23.3 Differentiate $y = \cos^5(x)$.

Write this as $y = \big(\cos(x)\big)^5$. Notice that this is a composition $\begin{cases} y = u^5 \\ u = \cos(x). \end{cases}$ Thus we should use the chain rule to get the derivative. Version 2 states

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 5u^4 \cdot \big(-\sin(x)\big) \\ &= 5\big(\cos(x)\big)^4 \cdot \big(-\sin(x)\big) && \text{(plug in } u = \cos(x)) \\ &= \boxed{-5\cos^4(x)\sin(x)} \end{aligned}$$

✍

Example 23.4 Differentiate $y = \sqrt{x^3 + 5x - 4}$.

This is a composition $\begin{cases} y = \sqrt{u} = u^{1/2} \\ u = x^3 + 5x - 4. \end{cases}$ By Version 2 of chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{2}u^{-1/2} \cdot \big(3x^2 + 5\big) \\ &= \frac{1}{2\sqrt{u}} \cdot \big(3x^2 + 5\big) \;=\; \boxed{\frac{3x^2 + 5}{2\sqrt{x^3 + 5x - 4}}} \end{aligned}$$

✍

The two examples above share a common structure. In the first case we differentiated $\big(\cos(x)\big)^5$ and in the second case we differentiated $(x^3 + 5x - 4)^{1/2}$. In both cases we differentiated an expression of the form $\big(g(x)\big)^n$. This pattern occurs so often that we will derive a general rule for it:

Example 23.5 Differentiate $y = \big(g(x)\big)^n$.

This is a composition $\begin{cases} y = u^n \\ u = g(x). \end{cases}$ By version 2 of chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= nu^{n-1} \cdot g'(x) \;=\; \boxed{n\big(g(x)\big)^{n-1} g'(x).} \end{aligned}$$

✍

Exercise 23.5 showed that $\dfrac{d}{dx}\Big[\big(g(x)\big)^n\Big] = n\big(g(x)\big)^{n-1} g'(x)$. This very useful formula is called the **generalized power rule**.

Generalized Power Rule: Given a real number n and a function $g(x)$,

$$\frac{d}{dx}\left[\left(g(x)\right)^n\right] = n\left(g(x)\right)^{n-1}g'(x).$$

Use this whenever you need to differentiate a function to a power.

Example 23.6 Differentiate $y = \left(x^4 + \sin(x) + 3\right)^7$.

Answer: $\frac{d}{dx}\left[\left(x^4+\sin(x)+3\right)^7\right] = \boxed{7\left(x^4+\sin(x)+3\right)^6\left(4x^3+\cos(x)\right)}$ ✍

Example 23.7 $\frac{d}{dx}\left[\frac{1}{x^2+1}\right] = \frac{d}{dx}\left[\left(x^2+1\right)^{-1}\right] = -(x^2+1)^{-2}2x = \boxed{\frac{-2x}{\left(x^2+1\right)^2}}.$

(You could also use the quotient rule.) ✍

Example 23.8 $\frac{d}{dx}\left[\tan^9(x)\right] = \frac{d}{dx}\left[\left(\tan(x)\right)^9\right] = \boxed{9\tan^8(x)\sec^2(x).}$ ✍

Of course you should expect to combine the generalized power rule with other rules, as the next example indicates.

Example 23.9 Find the derivative of $y = x^3 + 4\sin^3(x)$.

We'll do this step-by-step, but normally you will do a problem like this in your head, in one step.

$$\begin{aligned}
\frac{d}{dx}\left[x^3 + 4\sin^3(x)\right] &= \frac{d}{dx}\left[x^3\right] + \frac{d}{dx}\left[4\sin^3(x)\right] && \text{(sum-difference rule)}\\
&= \frac{d}{dx}\left[x^3\right] + 4\frac{d}{dx}\left[\sin^3(x)\right] && \text{(constant multiple rule)}\\
&= 3x^2 + 4\frac{d}{dx}\left[\sin^3(x)\right] && \text{(power rule)}\\
&= 3x^2 + 4\cdot 3\sin^2(x)\cos(x) && \text{(generalized power rule)}\\
&= \boxed{3x^2 + 12\sin^2(x)\cos(x).}
\end{aligned}$$

✍

Example 23.10 Find the derivative of $y = 4x^3\sin^3(x)$.

The is a product of two functions, so we must start with the product rule.

$$\begin{aligned}
\frac{d}{dx}\left[4x^3\sin^3(x)\right] &= \frac{d}{dx}\left[4x^3\right]\cdot\sin^3(x) + 4x^3\cdot\frac{d}{dx}\left[\sin^3(x)\right]\\
&= 12x^2\sin^3(x) + 4x^3\cdot 3\sin^2(x)\cos(x)\\
&= \boxed{12x^2\sin^3(x) + 12x^3\sin^2(x)\cos(x).}
\end{aligned}$$

✍

Example 23.11 Find the derivative of $y = e^{x^2+3x+1}$.

This is *not* a function to a power, so the generalized power rule is not the right tool. But it *is* a composition: $\begin{cases} y = e^u \\ u = x^2 + 3x + 1. \end{cases}$ Using the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot \left(2x+3\right) \\ &= \boxed{e^{x^2+3x+1} \cdot \left(2x+3\right)} \end{aligned}$$

✍

In Example 23.11 we differentiated $e^{g(x)}$ and got $e^{g(x)}g'(x)$. This pattern occurs often enough that we make a rule for it.

Generalized Exponential Rule: Given a function $g(x)$,

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}g'(x).$$

This rule is nothing but the chain rule combined with the rule $\frac{d}{dx}\left[e^x\right] = e^x$, just as the generalized power rule is nothing but the chain rule combined with the power rule. In that sense you don't even have to remember them. But they are very useful templates for a great many circumstances. Using them allows you to get an answer on one step, whereas setting up the problem with the chain rule and writing the details could take several steps.

Example 23.12 $\frac{d}{dx}\left[e^{\sqrt{x}}\right] = e^{\sqrt{x}}\frac{d}{dx}\left[\sqrt{x}\right] = e^{\sqrt{x}}\frac{1}{2\sqrt{x}} = \boxed{\frac{e^{\sqrt{x}}}{2\sqrt{x}}}$ ✍

Example 23.13 $\frac{d}{dx}\left[e^{-x}\right] = e^{-x}\frac{d}{dx}\left[-x\right] = e^{-x}(-1) = \boxed{-e^{-x}}$ ✍

Example 23.14 $\frac{d}{dx}\left[e^{\pi x}\right] = e^{\pi x}\frac{d}{dx}\left[\pi x\right] = e^{\pi x}\pi = \boxed{\pi e^{\pi x}}$ ✍

Example 23.15 $\frac{d}{dx}\left[e^{x\cos(x)}\right] = e^{x\cos(x)}\frac{d}{dx}\left[x\cos(x)\right] = \boxed{e^{x\cos(x)}\left(\cos(x) - x\sin(x)\right)}$
(We used the product rule in the final step.) ✍

With practice you will do problems like these in one step.

We've now seen examples of the generalized power rule and the generalized exponential rule. These are not the only such rules. In fact, each derivative rule for a specific function has a "chain rule generalization." Here is the list so far. The column on the left lists derivative rules $\frac{d}{dx}[f(x)]$ for specific functions $f(x)$. On the right are the corresponding chain rule statements $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

Rule $\frac{d}{dx}\Big[f(x)\Big]$	**Generalized rule** $\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big)\cdot g'(x)$
$\frac{d}{dx}\Big[x^n\Big] = nx^{n-1}$	$\frac{d}{dx}\Big[\big(g(x)\big)^n\Big] = n\big(g(x)\big)^{n-1}g'(x)$
$\frac{d}{dx}\Big[e^x\Big] = e^x$	$\frac{d}{dx}\Big[e^{g(x)}\Big] = e^{g(x)}\cdot g'(x)$
$\frac{d}{dx}\Big[\sin(x)\Big] = \cos(x)$	$\frac{d}{dx}\Big[\sin\big(g(x)\big)\Big] = \cos\big(g(x)\big)\cdot g'(x)$
$\frac{d}{dx}\Big[\tan(x)\Big] = \sec^2(x)$	$\frac{d}{dx}\Big[\tan\big(g(x)\big)\Big] = \sec^2\big(g(x)\big)\cdot g'(x)$
$\frac{d}{dx}\Big[\sec(x)\Big] = \sec(x)\tan(x)$	$\frac{d}{dx}\Big[\sec\big(g(x)\big)\Big] = \sec\big(g(x)\big)\tan\big(g(x)\big)\cdot g'(x)$
$\frac{d}{dx}\Big[\cos(x)\Big] = -\sin(x)$	$\frac{d}{dx}\Big[\cos\big(g(x)\big)\Big] = -\sin\big(g(x)\big)\cdot g'(x)$
$\frac{d}{dx}\Big[\cot(x)\Big] = -\csc^2(x)$	$\frac{d}{dx}\Big[\cot\big(g(x)\big)\Big] = -\csc^2\big(g(x)\big)\cdot g'(x)$
$\frac{d}{dx}\Big[\csc(x)\Big] = -\csc(x)\cot(x)$	$\frac{d}{dx}\Big[\csc\big(g(x)\big)\Big] = -\csc\big(g(x)\big)\cot\big(g(x)\big)\cdot g'(x)$

The generalized rules on the right are just the chain rule combined with specific rules you already know, so you don't really have to remember them. Even so, as you get used to using the chain rule you will find yourself conceptually combining steps so that you are in essence using the generalized rules as stated above. For example, if asked for the derivative of $\csc\big(x^3\big)$ you will immeidately spit out the answer $-\csc\big(x^3\big)\cot\big(x^3\big)3x^2$, not from memorizing the rule, but from pure understanding.

Example 23.16 Here are a two (deceptively?) simple ones.

$$\frac{d}{dx}\Big[\cos(x+1)\Big] = -\sin(x+1)\cdot 1 = \boxed{-\sin(x+1)}$$

$$\frac{d}{dx}\Big[\cot(-x)\Big] = -\csc^2(-x)\cdot(-1) = \boxed{\csc^2(-x)}$$

The structure of the chain rule is

$$\frac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'\big(g(x)\big)\cdot\frac{d}{dx}\Big[g(x)\Big].$$

In computing the $\frac{d}{dx}[g(x)]$ you use whichever rule applies. It can happen that $g(x)$ is itself a composition, in which case you use the chain rule for a second time in the same problem. Our final example illustrates this.

Example 23.17 Differentiate $y = \sec\big(\sin\big(x^3\big)\big)$.

$$\begin{aligned}\frac{d}{dx}\Big[\sec\Big(\sin\big(x^3\big)\Big)\Big] &= \sec\Big(\sin\big(x^3\big)\Big)\tan\Big(\sin\big(x^3\big)\Big)\frac{d}{dx}\Big[\sin\big(x^3\big)\Big] \\ &= \boxed{\sec\Big(\sin\big(x^3\big)\Big)\tan\Big(\sin\big(x^3\big)\Big)\cos\big(x^3\big)3x^2}\end{aligned}$$

In this example we used Version 1 of the chain rule. You can also use Version 2 for such a 3–tiered composition. Version 2 extends as follows.

.; **Version 2.1** If $y = f\big(g(h(x))\big)$, then $\begin{cases} y = f(u) \\ u = g(z) \\ z = h(x) \end{cases}$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dz}\cdot\dfrac{dz}{dx}$

So to differentiate $y = \sec\big(\sin\big(x^3\big)\big)$, we would write $\begin{cases} y = \sec(u) \\ u = \sin(z). \\ z = x^3 \end{cases}$ Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dz}\cdot\frac{dz}{dx} \\ &= \sec(u)\tan(u)\cdot\cos(z)\cdot 3x^2 \\ &= \sec\big(\sin(z)\big)\tan\big(\sin(z)\big)\cdot\cos(z)\cdot 3x^2 && (u = \sin(z)) \\ &= \boxed{\sec\Big(\sin\big(x^3\big)\Big)\tan\Big(\sin\big(x^3\big)\Big)\cos\big(x^3\big)3x^2} && (z = x^3)\end{aligned}$$

The chain rule can combine with the other derivative rules (and with itself) in a multitude of ways. It is vitally important to explore the possibilities by working lots of exercises. Do the odd-numbered ones and and check your work by looking at the solutions. (Or peek if you are really stuck.)

Exercises for Chapter 23

In exercises 1–36 find the derivative of the given function.

1. $\tan(\pi x)$ **2.** $\cos(x^2)$ **3.** e^{x^2+3} **4.** $e^{1+e^{-x}}$

5. $\dfrac{1}{z^2+3z}$ **6.** $\big(\cos(2x)\big)^2$ **7.** $\dfrac{1}{\big(\cos(2x)\big)^2}$ **8.** $\sqrt{3x+1}$

9. $(x^2+x)\sqrt{3x+1}$ **10.** $3\sec\left(\sqrt{x}\right)$ **11.** $\sqrt{4x^2+7x-3}$ **12.** $\sec\left(x^2e^x\right)$

13. $\dfrac{e^{\pi x}}{x^2+1}$ **14.** $x\sec\left(e^{10x}\right)$ **15.** $e^{x^2\sec(x)}$ **16.** $\sqrt{x^2+1}$

17. $\left(\dfrac{x^2+5}{x+1}\right)^4$ **18.** $\dfrac{x^2+1}{e^{\pi x}}$ **19.** $\sqrt{\dfrac{x^2+1}{e^x}}$ **20.** $\left(\dfrac{x^2}{e^x+1}\right)^{100}$

21. $\sec(x^2e^x)$ **22.** $e^{3x}\sqrt{x^4+1}$ **23.** $x^4\tan(\pi x)$ **24.** $x^3\sec(\pi x)$

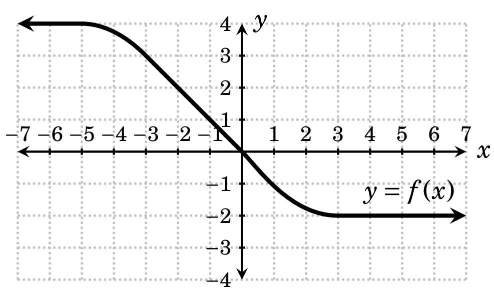
25. $e^{\sqrt{\cot(x)}}$ **26.** $xe^{\cos(3x)}$ **27.** $e^{4x}\sqrt{3x^2+x}$ **28.** $\left(x+\cos(x^2)\right)^9$

29. $\left(x^2+\sin(x)\right)^5$ **30.** $\left(x+e^{6x}\right)^9$ **31.** $\sqrt{\theta^5}+e+e^{\pi\theta}$ **32.** $\cos\left(\tan(x^3)\right)$

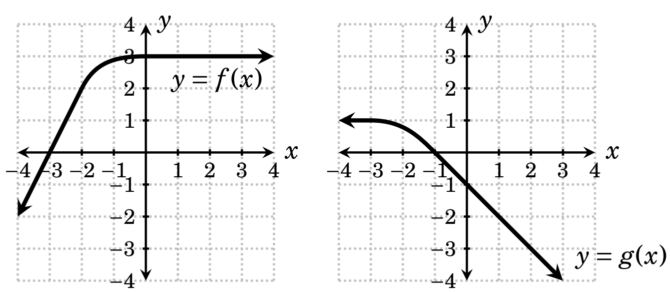
33. $x^2\cos^5(x)$ **34.** $\dfrac{x^2-4x}{e^{3x}}$ **35.** $\tan(x^5)+\tan^5(x)$ **36.** $\sec\left(e^{x^3+x}\right)$

37. Find all (x,y) on the graph of $y=\dfrac{1}{x-4}+x-4$ where the tangent is horizontal.

38. A function $f(x)$ is graphed below. Suppose $g(x)=(f(x))^5$. Find $g'(-2)$.



39. Two functions $f(x)$ and $g(x)$ are graphed below. Let $h(x)=f(g(x))$. Estimate $h'(2)$.



40. Information about functions $f(x)$, $g(x)$ and their derivatives is given in the table below. If $h(x) = f\big(g(x)\big)$, find $h'(3)$.

x	0	1	2	3	4	5
$f(x)$	−4	−2	0	1	1	0
$f'(x)$	2	1	1	3	0.5	−1
$g(x)$	10	9	7	4	0	−4
$g'(x)$	0	−0.5	−1	−3	−4	−4

41. Information about a function $f(x)$ and its derivative is given in the table below. If $h(x) = \big(f(x)\big)^4$, find $h'(2)$.

x	0	1	2	3	4	5
$f(x)$	0	−3	−2	3	10	25
$f'(x)$	−1	−7	−5	5	20	30

Exercise Solutions for Chapter 23

1. $\dfrac{d}{dx}\Big[\tan(\pi x)\Big] = \pi\sec^2(\pi x)$

3. $\frac{d}{dx}\Big[e^{x^2+3}\Big] = e^{x^2+3}2x$

5. $\dfrac{d}{dz}\Big[\dfrac{1}{z^2+3z}\Big] = \dfrac{d}{dz}\Big[\big(z^2+3z\big)^{-1}\Big] = -\big(z^2+3z\big)^{-2}(2z+3) = -\dfrac{2z+3}{\big(z^2+3z\big)^2}$

7. $\dfrac{d}{dx}\left[\dfrac{1}{\big(\cos(2x)\big)^2}\right] = \dfrac{d}{dx}\Big[\big(\cos(2x)\big)^{-2}\Big] = -2\big(\cos(2x)\big)^{-3}\dfrac{d}{dx}\Big[\cos(2x)\Big] = \dfrac{4\sin(2x)}{\cos^3(2x)}$

9. $\dfrac{d}{dx}\Big[(x^2+x)\sqrt{3x+1}\Big] = (2x+1)\sqrt{3x+1}+(x^2+x)\dfrac{3}{2\sqrt{3x+1}} = (2x+1)\sqrt{3x+1}+\dfrac{3x^2+3x}{2\sqrt{3x+1}}$

11. $\dfrac{d}{dx}\Big[\sqrt{4x^2+7x-3}\Big] = \dfrac{d}{dx}\Big[\big(4x^2+7x-3\big)^{1/2}\Big] = \dfrac{1}{2}\big(4x^2+7x-3\big)^{-1/2}(8x+7) = \dfrac{8x+7}{2\sqrt{4x^2+7x-3}}$

13. $\dfrac{d}{dx}\Big[\dfrac{e^{\pi x}}{x^2+1}\Big] = \dfrac{e^{\pi x}\pi\big(x^2+1\big)-e^{\pi x}2x}{\big(x^2+1\big)^2} = e^{\pi x}\dfrac{\pi x^2-2x+\pi}{\big(x^2+1\big)^2}$

15. $\dfrac{d}{dx}\Big[e^{x^2\sec(x)}\Big] = e^{x^2\sec(x)}\dfrac{d}{dx}\Big[x^2\sec(x)\Big] = e^{x^2\sec(x)}\big(2x\sec(x)+x^2\sec(x)\tan(x)\big)$

17. $\dfrac{d}{dx}\left[\left(\dfrac{x^2+5}{x+1}\right)^4\right] = 4\left(\dfrac{x^2+5}{x+1}\right)^3\dfrac{2x(x+1)-(x^2+5)\cdot 1}{(x+1)^2} = 4\left(\dfrac{x^2+5}{x+1}\right)^3\dfrac{x^2+2x-5}{(x+1)^2}$

19. $\dfrac{d}{dx}\left[\sqrt{\dfrac{x^2+1}{e^x}}\right] = \dfrac{d}{dx}\left[\left(\dfrac{x^2+1}{e^x}\right)^{1/2}\right] = \dfrac{1}{2}\left(\dfrac{x^2+1}{e^x}\right)^{-1/2}\dfrac{2xe^x-(x^2+1)e^x}{(e^x)^2} = \sqrt{\dfrac{e^x}{x^2+1}}\,\dfrac{2x-x^2-1}{2e^x}$

21. $\dfrac{d}{dx}\Big[\sec(x^2e^x)\Big] = \sec(x^2e^x)\tan(x^2e^x)\big(2xe^x+x^2e^x\big)$

23. $\dfrac{d}{dx}\Big[x^4\tan(\pi x)\Big] = 4x^3\tan(\pi x)+x^4\sec^2(\pi x)\pi$

25. $\dfrac{d}{dx}\Big[e^{\sqrt{\cot(x)}}\Big] = e^{\sqrt{\cot(x)}}\dfrac{d}{dx}\Big[\sqrt{\cot(x)}\Big] = e^{\sqrt{\cot(x)}}\dfrac{d}{dx}\Big[\big(\cot(x)\big)^{1/2}\Big] =$

$$e^{\sqrt{\cot(x)}}\frac{1}{2}\big(\cot(x)\big)^{-1/2}\big(-\csc^2(x)\big) = -\frac{e^{\sqrt{\cot(x)}}\csc^2(x)}{2\sqrt{\cot(x)}}$$

27. $\dfrac{d}{dx}\Big[e^{4x}\sqrt{3x^2+x}\Big] = e^{4x}4\sqrt{3x^2+x}+e^{4x}\dfrac{d}{dx}\Big[\sqrt{3x^2+x}\Big] = e^{4x}4\sqrt{3x^2+x}+e^{4x}\dfrac{6x+1}{2\sqrt{3x^2+x}}$

$$= e^{4x}\left(4\sqrt{3x^2+x}+\frac{6x+1}{2\sqrt{3x^2+x}}\right)$$

29. $\dfrac{d}{dx}\Big[\big(x^2+\sin(x)\big)^5\Big] = 5\big(x^2+\sin(x)\big)^4\big(2x+\cos(x)\big)$

31. $\dfrac{d}{d\theta}\Big[\sqrt{\theta^5}+e+e^{\pi\theta}\Big] = \dfrac{d}{d\theta}\Big[\theta^{5/2}+e+e^{\pi\theta}\Big] = \dfrac{5}{2}\theta^{5/2-1}+0+e^{\pi\theta}\pi = \dfrac{5}{2}\sqrt{\theta^3}+\pi e^{\pi\theta}$

33. $\dfrac{d}{dx}\left[x^2\cos^5(x)\right] = 2x\cos^5(x) + x^2 5\cos^4(x)(-\sin(x)) = 2x\cos^5(x) - 5x^2\cos^4(x)\sin(x)$

35. $\dfrac{d}{dx}\left[\tan\left(x^5\right) + \tan^5(x)\right] = \sec^2\left(x^5\right)5x^4 + 5\tan^4(x)\sec^2(x)$

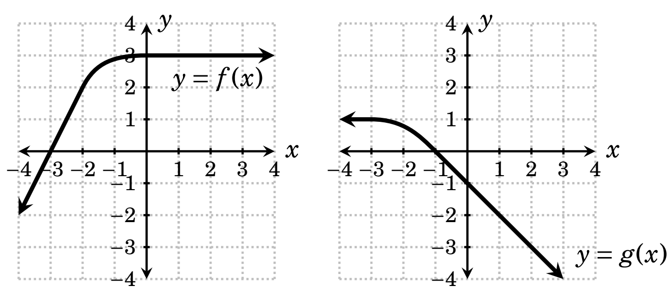
37. Find all (x, y) on the graph of $y = \dfrac{1}{x-4} + x - 4$ where the tangent is horizontal.

Write this as $y = (x-4)^{-1} + x - 4$. Then $\dfrac{dy}{dx} = -(x-4)^{-2} + 1 = \dfrac{-1}{(x-4)^2} + 1$. To find where the tangent is horizontal, we set this equal to 0 and solve.

$$\begin{aligned}\frac{-1}{(x-4)^2} + 1 &= 0\\ 1 &= \frac{1}{(x-4)^2}\\ (x-4)^2 &= 1\\ x-4 &= \pm 1\\ x &= 4 \pm 1\end{aligned}$$

Thus the tangent is horizontal when $x = 3$ and $x = 5$. The points of tangency are thus $(3, -2)$ and $(5, 2)$.

39. Two functions $f(x)$ and $g(x)$ are graphed below. Let $h(x) = f(g(x))$. Estimate $h'(2)$.



By the chain rule, $h'(x) = f'\big(g(x)\big) \cdot g'(x)$. Reading information off the graphs, $h'(2) = f'\big(g(2)\big) \cdot g'(2) = f'(-3) \cdot (-1) = 2 \cdot (-1) = \boxed{-2.}$

41. Information about a function $f(x)$ and its derivative is given in the table below. If $h(x) = \big(f(x)\big)^4$, find $h'(2)$.

x	0	1	2	3	4	5
$f(x)$	0	−3	−2	3	10	25
$f'(x)$	−1	−7	−5	5	20	30

By the generalized power rule, $h'(x) = 4\big(f(x)\big)^3 \cdot f'(x)$. Reading information off the chart, we get $h'(2) = 4\big(f(2)\big)^3 \cdot f'(2) = 4\big(-2\big)^3 \cdot (-5) = \boxed{160.}$