We talk in terms of relations, we relate items, objects, …, etc

• Examples:
  • is bigger than,
  • is more expensive than,
  • is parallel to,
  • is a subset of
  • is ….

• All these expression relate some ‘items’, or they express the existence or nonexistence of a certain connection between a pair of objects taken in a definite order

Relations

• If we want to describe a relationship between elements of two sets A and B, we can use ordered pairs with their first element taken from A and their second element taken from B.
• Since this is a relation between two sets, it is called a binary relation.

• Definition: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$. (Hence, A BINARY RELATION IS A SET of PAIRS).

• In other words, for a binary relation $R$ we have $R \subseteq A \times B$. We use the notation $aRb$ to denote that $(a, b) \in R$ and $aRb$ to denote that $(a, b) \notin R$.
And, what would be a Cartesian product $P \times C$?

- $P \times C = \{(\text{Carl, Mercedes})*, (\text{Carl, BMW}), (\text{Carl, tricycle}), (\text{Suzanne, Mercedes})*, (\text{Suzanne, BMW})*, (\text{Suzanne, tricycle})$, $(\text{Peter, Mercedes}), (\text{Peter, BMW}), (\text{Peter, tricycle})*, (\text{Carla, Mercedes}), (\text{Carla,BMW}), (\text{Carla, tricycle})\}$

  OBVIOUSLY!!!

$$pRc \subseteq P \times C$$

Functions as Relations

- You might remember that a function $f$ from a set $A$ to a set $B$ assigns a unique element of $B$ to each element of $A$.
- The graph of $f$ is the set of ordered pairs $(a, b)$ such that $b = f(a)$.
- Since the graph of $f$ is a subset of $A \times B$, it is a relation from $A$ to $B$.
- Moreover, for each element $a$ of $A$, there is exactly one ordered pair in the graph that has $a$ as its first element.

Functions as Relations

- Conversely, if $R$ is a relation from $A$ to $B$ such that every element in $A$ is the first element of exactly one ordered pair of $R$, then a function can be defined with $R$ as its graph.
- This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

Thus,

- Relations are a generalization of functions and they can be used to express a much wider class of relationships between sets.
Relations on a Set

• **Definition:** A relation on the set $A$ is a relation from $A$ to $A$.

• In other words, a relation on the set $A$ is a subset of $A \times A$.

• **Example:** Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a < b\}$?

We can show this relation graphically over 2-dim space.

We can also state

If $R$ is a relation from a set $A$ to itself, that is, if $R$ is a subset of $A^2 = A \times A$, then we say that $R$ is a relation on $A$.

The **domain** of a relation $R$ is the set of all first elements of the ordered pairs which belong to $R$, and the **range** of $R$ is the set of second elements.
Examples:

(a) Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$, and let $R = \{(1, x), (1, z), (3, y)\}$. Then $R$ is a relation from $A$ to $B$ since $R$ is a subset of $A \times B$ with respect to this relation.

(b) Let $A = \{\text{eggs, milk, corn}\}$ and $B = \{\text{cows, goats, hens}\}$. We can define a relation $R$ from $A$ to $B$ by $(a, b) \in R$ if $a$ is produced by $b$. In other words,

$$R = \{(\text{eggs, hens}), (\text{milk, cows}), (\text{milk, goats})\}$$

With respect to this relation,

$$\text{eggs} \not\in R \quad \text{milk} \in R \text{ cows, etc.}$$

(c) Suppose we say that two countries are adjacent if they have some part of their boundaries in common. Then “is adjacent to” is a relation $R$ on the countries of the earth. Thus

$$(\text{Italy, Switzerland}) \in R \quad \text{but} \quad (\text{Canada, Mexico}) \not\in R$$

Properties of Relations

- **Definition:** A relation $R$ on a set $A$ is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

- Are the following relations on $\{1, 2, 3, 4\}$ reflexive?

  - $R = \{(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)\}$
    - No.
  - $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)\}$
    - Yes.
  - $R = \{(1, 1), (2, 2), (3, 3)\}$
    - No.

- **Definition:** A relation on a set $A$ is called **irreflexive** if $(a, a) \not\in R$ for every element $a \in A$.

More examples

Consider the following five relations:

1. Relation $\leq$ (less than or equal) on the set $\mathbb{Z}$ of integers
2. Set inclusion $\subseteq$ on a collection $\mathcal{C}$ of sets
3. Relation $\perp$ (perpendicular on the set $L$ of lines in the plane.
4. Relation $\parallel$ (parallel) on the set $L$ of lines in the plane.
5. Relation $|\text{ of divisibility on the set } \mathbb{N}$ of positive integers. (Recall \(x \mid y\) if there exists $z$ such that $xz = y$.)

Determine which of the relations are reflexive.

Solutions:

The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every integer $x$ in $\mathbb{Z}$, $\mathcal{C} \subseteq \mathcal{C}$ for any set $\mathcal{C}$ in $\mathcal{C}$, and $x \mid x$ for every positive integer $n$ in $\mathbb{N}$.
Properties of Relations

•Definitions:

• A relation \( R \) on a set \( A \) is called **symmetric** if \((b, a) \in R\) whenever \((a, b) \in R\) for all \(a, b \in A\).

• A relation \( R \) on a set \( A \) is called **antisymmetric** if whenever \((a, b) \in R\), \((b, a) \notin R\).

• A relation \( R \) on a set \( A \) such that for all \(a, b \in A\), if \((a, b) \in R\) and \((b, a) \in R\), then \(a = b\) is **antisymmetric**.

• A relation \( R \) on a set \( A \) is called **asymmetric** if \((a, b) \in R\) implies that \((b, a) \notin R\) for some \(a, b \in A\).

\( R \) can be both symmetric and antisymmetric, but it can’t be symmetric and asymmetric.

Examples: Which relations are symmetric

Consider the following five relations on the set \( A = \{1, 2, 3, 4\}\):

- \( R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}\)
- \( R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}\)
- \( R_3 = \{(1, 3), (2, 1)\}\)
- \( R_4 = \emptyset\), the empty relation
- \( R_5 = A \times A\), the universal relation

Solutions:

\( R_1 \) is not symmetric since \((1, 2) \in R_1\) but \((2, 1) \notin R_1\). \( R_2 \) is not symmetric since \((1, 3) \in R_2\) but \((3, 1) \notin R_2\). The other relations are symmetric.

Properties of Relations

• Are the following relations on \( \{1, 2, 3, 4\} \) symmetric, antisymmetric, or asymmetric?

- \( R = \{(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)\} \)  
  • symmetric
- \( R = \{(1, 1)\} \)  
  • sym. and antisym.
- \( R = \{(1, 3), (3, 2), (2, 1)\} \)  
  • antisym. and asym.
- \( R = \{(1, 3), (3, 2), (2, 1), (3, 1)\} \)  
  • asym.
- \( R = \{(4, 4), (3, 3), (1, 4)\} \)  
  • antisym.

Properties of Relations

• Definition: A relation \( R \) on a set \( A \) is called **transitive** if whenever \((a, b) \in R\) and \((b, c) \in R\), then \((a, c) \in R\) for \(a, b, c \in A\).

• Are the following relations on \( \{1, 2, 3, 4\} \) transitive?

- \( R = \{(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)\} \)  
  • Yes.
- \( R = \{(1, 3), (3, 2), (2, 1)\} \)  
  • No.
- \( R = \{(1, 3), (3, 2), (1, 2)\} \)  
  • Yes.
- \( R = \{(2, 4), (4, 3), (2, 3), (4, 1)\} \)  
  • No.
Some relations’ summary

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<th>&gt;</th>
<th>≤</th>
<th>≥</th>
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<td></td>
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<td>X</td>
<td>X</td>
<td>X</td>
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</table>

Basics of Counting
Permutations and Combinations

- Permutations - Order matters
- Combinations – Order doesn’t matter

<table>
<thead>
<tr>
<th>Combination</th>
<th>Permutations</th>
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<tr>
<td>abc</td>
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<tr>
<td>abd</td>
<td>abd, adb, bad, bda, dab, dba</td>
</tr>
<tr>
<td>acd</td>
<td>acd, adc, cad, cda, dac, dca</td>
</tr>
<tr>
<td>bcd</td>
<td>bcd, bdc, cdb, cdb, dbc, dcba</td>
</tr>
</tbody>
</table>

Example 1: How many words we can make from letters – H, O, W
HOW, HWO, OHW, OWH, WHO, WOH
- In languages – order matters

Example 2: How many words we can make from letters – H, O, W by using 2 letters only
HO, OH, HW, WH, OW, WO

Example 3: How many SETS of 3 letters we can make from letters – H, O, W
HOW, => 1 SET only

Example 4: How many SETS OF 2 LETTERS we can make from letters – H, O, W
HO, HW, OW

Counting permutations and combinations

- Permutations: \( P_r^n = \frac{n!}{(n-r)!} \)  
  Note: \( P_r^n = \frac{n!}{(n-r)!} = \frac{n!}{0!} = n! \)
  r terms

- Combinations: \( C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots}{1 \cdot 2 \cdot 3 \cdots r} \)

\( C_4^8 = \binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70 \)

Examples:

Note: \( \binom{n}{r} = \binom{n}{n-r} \), eg. \( \binom{8}{3} = \binom{8}{5} \),
Note also: \( \binom{n}{0} = \binom{n}{n} = 1 \), eg. \( \binom{8}{0} = \binom{8}{8} = 1 \)
Relations on a Set

• How many different relations can we define on a set A with n elements?

• A relation on a set A is a subset of A × A.
• How many elements are in A × A?

• There are n^2 elements in A × A, so how many subsets (= relations on A) does A × A have?

• The number of subsets that we can form out of a set with m elements is 2^m. Therefore, 2^{n^2} subsets can be formed out of A × A.

• Answer: We can define 2^{n^2} different relations on A.

Example of Relations Counting

• Show all the different relations on a set A = {a, b}?

• Solution: Relations on A are subsets of A × A, which contains 2^2 = elements as follows {(a, a), (a, b), (b, a), (b, b)}. Therefore, different relations on A can be generated by choosing different subsets out of these 4 elements, so there are 2^4 = 16 relations as follows:

1. {∅}, 2. {(a, a)}, 3. {(a, b)}, 4. {(b, a)}, 5. {(b, b)}, 6. {(a, a), (a, b)}, 7. {(a, a), (b, a)}, 8. {(a, a), (b, b)}, 9. {(a, b), (b, a)}, 10. {(a, b), (b, b)}, 11. {(b, a), (b, b)}, 12. {(a, a), (a, b), (b, a)}, 13. {(a, a), (a, b), (b, b)}, 14. {(a, b), (a, a), (b, b)}, 15. {(a, b), (a, b), (b, b)}, 16. {(a, a), (a, b), (b, a), (b, b)}

Let’s check the claim about 2^m

Set A={a, b, c}

How many subsets of A can we create?
1. {∅}
2. {a}
3. {b}
4. {c}
5. {a, b}
6. {a, c}
7. {b, c}
8. {a, b, c}

How can we, in a clear and clean way, find out this number?
Well, it’s basically the answer to the question – how many combinations of 0th, 1st, 2nd, 3rd, …, m-th order can we make from m elements of the set A?

Number of subsets = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{m}

In a particular example above, m = 3 and

Number of subsets = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8

Example:

Find the number of relations from A = {a, b, c} to B = {1, 2}. 

Counting Relations

• **Example:** How many different **reflexive** relations can be defined on a set A containing n elements?

• **Solution:** Relations on R are subsets of A×A, which contains n² elements.
  • Therefore, different relations on A can be generated by choosing different subsets out of these n² elements, so there are 2ⁿ² relations.
  • A reflexive relation, however, must contain the n elements (a, a) for every a ∈ A.
  • Consequently, we can only choose among n² – n = n(n – 1) elements to generate reflexive relations, so there are 2ⁿ(ⁿ⁻¹) of them.

Let’s check the claim about 2ⁿ(ⁿ⁻¹)

Set A={a, b}

\[ 2^{2(2-1)} = 2^2 = 4 \]

1. {0}, 2. {(a, a)}, 3. {(a, b)}, 4. {(b, a)}, 5. {(b, b)},
6. {(a, a), (a, b)}, 7. {(a, a), (b, a)}, 8. {(a, a), (b, b)},
9. {(a, b), (b, a)}, 10. {(a, b), (b, b)}, 11. {(b, a), (b, b)},
12. {(a, a), (a, b), (b, a)}, 13. {(a, a), (a, b), (b, b)},
14. {(a, a), (b, a), (b, b)}, 15. {(a, b), (b, a), (b, b)},
16. {(a, a), (a, b), (b, a), (b, b)}

Combining Relations

• Relations are sets, and therefore, we can apply the usual set operations to them.

• If we have two relations R₁ and R₂, and both of them are from a set A to a set B, then we can combine them to R₁ ∪ R₂, R₁ ∩ R₂, or R₁ – R₂.

• In each case, the result will be another relation from A to B.

Combining Relations

• … and there is another important way to combine relations.

• **Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where a ∈ A, c ∈ C, and for which there exists an element b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. We denote the composite of R and S by S•R.

• In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then S•R contains a pair (a, c).
Combining Relations

- **Example:** Let D and S be relations on $A = \{1, 2, 3, 4\}$.
  - $D = \{(a, b) \mid b = 5 - a\}$  "b equals (5 – a)"
  - $S = \{(b, c) \mid b < c\}$  "b is smaller than c"
  - $D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
  - $S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
  - $S \circ D = \{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

- D maps an element $a$ to the element $(5 – a)$, and afterwards S maps $(5 – a)$ to all elements larger than $(5 – a)$, resulting in $S \circ D = \{(a, c) \mid c > 5 – a\}$.

Combining Relations

- **Definition:** Let $R$ be a relation on the set $A$. The powers $R^n$, $n = 1, 2, 3, \ldots$, are defined inductively by
  - $R^1 = R$
  - $R^{n+1} = R^n \circ R$

- In other words:
  - $R^n = R \circ R \circ \ldots \circ R$ (n times the letter $R$)

Combining Relations

- **Theorem:** The relation $R$ on a set $A$ is transitive if and only if $R^n \subseteq R$ for all positive integers $n$.

- **Definition:** A relation $R$ on a set $A$ is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

- The composite of $R$ with itself contains exactly these pairs $(a, c)$.
- Therefore, for a transitive relation $R$, $R \circ R$ does not contain any pairs that are not in $R$, so $R \circ R \subseteq R$.
- Since $R \circ R$ does not introduce any pairs that are not already in $R$, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$. 

Combining Relations

- We already know that functions are just special cases of relations (namely those that map each element in the domain onto exactly one element in the codomain).

- If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).
Examples: Which relations are transitive?

Consider the following five relations on the set \( A = \{1, 2, 3, 4\} \):

- \( R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\} \)
- \( R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\} \)
- \( R_3 = \{(1, 3), (2, 1)\} \)
- \( R_4 = \emptyset \), the empty relation
- \( R_5 = A \times A \), the universal relation

Solutions:

The relation \( R_3 \) is not transitive since \( (2, 1), (1, 3) \in R_3 \) but \( (2, 3) \notin R_3 \). All the other relations are transitive.

n-ary Relations

• In order to study an interesting application of relations, namely databases, we first need to generalize the concept of binary relations to n-ary relations.

• Definition: Let \( A_1, A_2, \ldots, A_n \) be sets. An n-ary relation on these sets is a subset of \( A_1 \times A_2 \times \cdots \times A_n \).

• The sets \( A_1, A_2, \ldots, A_n \) are called the domains of the relation, and \( n \) is called its degree.

Examples: Which relations are transitive?

1. Relation \( \leq \) (less than or equal) on the set \( \mathbb{Z} \) of integers
2. Set inclusion \( \subseteq \) on a collection \( \mathcal{C} \) of sets
3. Relation \( \perp \) (perpendicular) on the set \( L \) of lines in the plane.
4. Relation \( \parallel \) (parallel) on the set \( L \) of lines in the plane.
5. Relation \( | \) (divisibility) on the set \( \mathbb{N} \) of positive integers. (Recall \( x|y \) if there exists \( z \) such that \( xz = y \).)

Solutions:

The relations \( \leq, \subseteq, \) and \( \parallel \) are transitive. That is: (i) If \( a \leq b \) and \( b \leq c \), then \( a \leq c \). (ii) If \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \). (iii) If \( a \parallel b \) and \( b \parallel c \), then \( a \parallel c \).

On the other hand the relation \( \perp \) is not transitive. If \( a \perp b \) and \( b \perp c \), then it is not true that \( a \perp c \). Since no line is parallel to itself, we can have \( a \parallel b \) and \( b \parallel a \), but \( a \not\parallel a \). Thus \( \parallel \) is not transitive. (We note that the relation “is parallel or equal to” is a transitive relation on the set \( L \) of lines in the plane.)
Let us take a look at a type of database representation that is based on relations, namely the relational data model.

A database consists of n-tuples called records, which are made up of fields.

These fields are the entries of the n-tuples.

The relational data model represents a database as an n-ary relation, that is, a set of records.

Although not all tables are relations, the terms table and relation are normally used interchangeably.

The following sets of terms are equivalent:

Databases and Relations

- Example: Consider a database of students, whose records are represented as 4-tuples with the fields Student Name, ID Number, Major, and GPA:

\[ R = \{(\text{Ackermann}, 231455, \text{CS}, 3.88), \]
\( (\text{Adams}, 888323, \text{Physics}, 3.45), \)
\( (\text{Chou}, 102147, \text{CS}, 3.79), \)
\( (\text{Goodfriend}, 453876, \text{Math}, 3.45), \)
\( (\text{Rao}, 678543, \text{Math}, 3.90), \)
\( (\text{Stevens}, 786576, \text{Psych}, 2.99)\}\)

Relations that represent databases are also called tables, since they are often displayed as tables.

A domain of an n-ary relation is called a primary key if the n-tuples are uniquely determined by their values from this domain.

This means that no two records have the same value from the same primary key.

In our example, which of the fields Student Name, ID Number, Major, and GPA are primary keys?

Student Name and ID Number are primary keys, because no two students have identical values in these fields.

In a real student database, only ID Number would be a primary key.
• In a database, a **primary key should remain same** even if new records are added.

• **Combinations of domains** can also uniquely identify n-tuples in an n-ary relation.

• When the values of a **set of domains** determine an n-tuple in a relation, the **Cartesian product** of these domains is called a **composite key**.

---

**Keys**

• A **key** is a combination of one or more columns that is used to identify rows in a relation

• A **composite key** is a key that consists of two or more columns

---

**Candidate and Primary Keys**

• A **candidate key** is a key that **determines all of the other columns** in a relation

• A **primary key** is a candidate key selected as the **primary means of identifying rows** in a relation:
  - There is one and only one primary key per relation
  - The primary key may be a composite key
  - The ideal primary key is short, numeric and never changes

---

**We can apply a variety of operations on n-ary relations to form new relations.**

• **Definition**: The projection \( P_{i_1, i_2, \ldots, i_m} \) maps the n-tuple \((a_1, a_2, \ldots, a_n)\) to the m-tuple \((a_{i_1}, a_{i_2}, \ldots, a_{i_m})\), where \( m \leq n \).

• In other words, a projection \( P_{i_1, i_2, \ldots, i_m} \) keeps the m components \( a_{i_1}, a_{i_2}, \ldots, a_{i_m} \) of an n-tuple and deletes its \((n - m)\) other components.

• **Example**: What is the result when we apply the projection \( P_{2,4} \) to the student record (Stevens, 786576, Psych, 2.99) ?

• **Solution**: It is the pair (786576, 2.99).
In some cases, applying a projection to an entire table may not only result in fewer columns, but also in fewer rows.

Why is that?

Some records may only have differed in those fields that were deleted, so they become identical, and there is no need to list identical records more than once.

We can use the join operation to combine two tables into one if they share some identical fields.

Definition: Let R be a relation of degree m and S a relation of degree n. The join $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all ($m + n - p$)-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

In other words, to generate $J_p(R, S)$, we have to find all the elements in R whose p last components match the p first components of an element in S.

The new relation contains exactly these matches, which are combined to tuples that contain each matching field only once.

Example: What is $J_1(Y, R)$, where Y contains the fields Student Name and Year of Birth, and R contains the student records as defined before?

- and R contains the student records as defined before?
Databases and Relations

**Solution:** The resulting relation is:

- \{(1978, Ackermann, 231455, CS, 3.88),
  (1972, Adams, 888323, Physics, 3.45),
  (1917, Chou, 102147, CS, 3.79),
  (1984, Goodfriend, 453876, Math, 3.45),
  (1982, Rao, 678543, Math, 3.90),
  (1970, Stevens, 786576, Psych, 2.99)\}

Since \(Y\) has two fields and \(R\) has four, the relation \(J_1(Y, R)\) has \(2 + 4 - 1 = 5\) fields.