1. (15 points) A graph $G$ is drawn below. Label each vertex with its eccentricity. State the radius and diameter of $G$. Indicate the center of $G$.

Radius is 3.
Diameter is 6.
The center is the single shaded vertex with minimum eccentricity 3.

2. (15 points) Suppose $k \geq 2$. Prove that a $k$-regular bipartite graph has no cut-edge.

**Proof.** Suppose for the sake of contradiction that $G$ is a $k$-regular bipartite graph ($k \geq 2$) with a cut edge $ab$. When $ab$ is removed from $G$, the component of $G$ containing the edge $ab$ splits into two new components; call them $A$ and $B$, with $a \in A$ and $b \in B$. Both of these components are nontrivial, since their vertices have degrees at least $k - 1 \geq 1$. Now, the component $A$ is bipartite (since it is a subgraph of a bipartite graph), so there is a bipartition $V(A) = X \cup Y$ of $A$ with each edge of $A$ running between $X$ and $Y$. Without loss of generality, say $a \in Y$. Then every vertex of $X$ has degree $k$. By contrast every vertex of $Y$ has degree $k$, except for the vertex $a$, which has degree $k - 1$. Therefore, we can count the number of edges in $A$ in two ways:

$$|E(A)| = k|X| = k(|Y| - 1) + (k - 1)$$
$$k|X| = k|Y| - 1$$
$$1 = k(|Y| - |X|)$$
$$\frac{1}{k} = |Y| - |X| \in \mathbb{Z}.$$

From the above, it follows that $k = 1$, contradicting the fact that $k \geq 2$. QED
3. (15 points) Let \( k \geq 2 \) be a fixed integer. Suppose a tree \( T \) has \( p \) vertices of degree \( k \), and all the other vertices of \( T \) have degree 1. Find \( n(T) \).

**Proof.** Since \( T \) has \( p \) vertices of degree \( k \) and \( n(T) - p \) vertices of degree 1, we have

\[
2|E(T)| = \sum_{x \in V(T)} d(x) = p \cdot k + (n(T) - p) \cdot 1 = p(k - 1) + n(T).
\]

But \( T \) is a tree, so \( |E(T)| = n(T) - 1 \), and the above calculation yields

\[
2(n(T) - 1) = p(k - 1) + n(T) \Rightarrow 2n(T) - 2 = p(k - 1) + n(T) \Rightarrow n(T) = p(k - 1) + 2.
\]

Therefore \( n(T) = p(k - 1) + 2 \).

4. (15 points) State the following theorems carefully and precisely.

(a) Berge’s Theorem

A matching \( M \) in a graph \( G \) is a maximum matching if and only if \( G \) has no \( M \)-augmenting path.

(An \( M \)-augmenting path is a path which alternates between edges in \( M \) and not in \( M \), and whose endpoints are not saturated by \( M \).)

(b) Hall’s Theorem:

Suppose \( G \) is a bipartite graph with bipartition \( V(G) = X \cup Y \).

Then \( G \) has a matching that saturates \( X \) if and only if \( |N(S)| \geq |S| \) for all \( S \subseteq X \).

(Here \( N(S) \) denotes the set of all vertices of \( G \) which are adjacent to a vertex of \( S \).)

(c) The König-Egerváry Theorem

For any bipartite graph \( G \), the maximum size of a matching equals the minimum size of a vertex cover.

(A vertex cover is a set \( Q \subseteq V(G) \) such that every edge of \( G \) has an endpoint in \( Q \).)
5. (20 points) Find the listed invariants for the Petersen graph.

(a) $\alpha = 4$ ($\{2, 0, 8, 9\}$ is a maximum independent set.)

(b) $\gamma = 3$ ($\{3, 5, 6\}$ is a minimum dominating set.)

(c) $\alpha' = 5$ ($\{05, 16, 27, 38, 49\}$ is a perfect matching.)

(d) $\chi = 3$ ($\chi > 2$ since $P$ has 5-cycle; See 3-coloring on left.)

(e) $\omega = 2$ (Petersen graph has $K_2$'s but no $K_3$'s.)

6. (10 points) Prove that $\gamma \leq \alpha$ for any graph.

**Proof.** Let $I$ be a largest independent set in $G$, so $|I| = \alpha$. Now, if $x$ is any vertex of $G$, then either $x \in I$, or $x$ is adjacent to a vertex in $I$. (If $x$ were not adjacent to a vertex in $I$, then we could enlarge the independent set $I$ by appending $x$ to it, but $I$ is already a largest independent set. Since every vertex of $G$ is in either $I$ or adjacent to a vertex in $I$, it follows that $I$ is a dominating set. Since $\gamma$ is the size of a smallest dominating set, we have $\gamma \leq I = \alpha$. QED

7. (10 points) Prove that $\chi \cdot \alpha \geq n$ for any graph.

**Proof.** Consider a coloring of $G$ with $\chi$ colors $\{1, 2, \ldots, \chi\}$. For any color $i$, the set $X_i$ of vertices with that color is an independent set in $G$, and therefore $|X_i| \leq \alpha$. Therefore we have

$$n = |X_1| + |X_2| + \cdots + |X_\chi|$$

$$\leq \alpha + \alpha + \cdots + \alpha = \chi \alpha.$$

This establishes $\chi \cdot \alpha \geq n$. QED