while if \( n = 4 \) they are

\[
\begin{array}{cccc}
4 & 1 & 3 & 2 \\
4 & 3 & 2 & 1 \\
3 & 2 & 1 & 4 \\
2 & 4 & 3 & 1 \\
1 & 3 & 2 & 4
\end{array}
\]

Thus \( Q_1 = 1 \), \( Q_2 = 1 \), \( Q_3 = 3 \), and \( Q_4 = 11 \).

**Theorem 6.5.1** For \( n \geq 1 \)

\[
Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}1!
\]

**Proof.** Let \( S \) be the set of all \( n! \) permutations of \( \{1, 2, \ldots, n\} \). Let \( P_j \) be the property that in a permutation the pattern \( j(j+1) \) does occur, \( j = 1, 2, \ldots, n-1 \). Thus a permutation of \( \{1, 2, \ldots, n\} \) is counted in the number \( Q_n \) if and only if it has none of the properties \( P_1, P_2, \ldots, P_{n-1} \). As usual let \( A_j \) denote the set of permutations of \( \{1, 2, \ldots, n\} \) which satisfy property \( P_j \), \( j = 1, 2, \ldots, n-1 \). Then

\[
Q_n = |A_1 \cap A_2 \cap \cdots \cap A_{n-1}|
\]

and we apply the inclusion-exclusion principle to evaluate \( Q_n \). We first calculate the number of permutations in \( A_1 \). A permutation is in \( A_1 \) if and only if the pattern 12 occurs in it. Thus a permutation in \( A_1 \) may be regarded as a permutation of the \( n-1 \) symbols \( \{12, 3, 4, \ldots, n\} \). We conclude that \( |A_1| = (n-1)! \), and in general we see that

\[
|A_j| = (n-1)! \quad (j = 1, 2, \ldots, n-1).
\]

Permutations which are in two of the sets \( A_1, A_2, \ldots, A_{n-1} \) contain two patterns. These patterns either share an element, like the patterns 12 and 23 or have no element in common, like the patterns 12 and 34. A permutation which contains the two patterns 12 and 34 can be regarded as a permutation of the \( n-2 \) symbols \( \{12, 34, 5, \ldots, n\} \). Thus \( |A_1 \cap A_3| = (n-2)! \). A permutation which contains the two patterns 12 and 23 contains the pattern \( 123 \) and thus can be regarded as a permutation of the \( n-2 \) symbols \( \{12, 3, 4, \ldots, n\} \). Thus \( |A_1 \cap A_2| = (n-2)! \). In general, we see that

\[
|A_1 \cap A_j| = (n-2)!\]

for each 2-combination \( \{i, j\} \) of \( \{1, 2, \ldots, n-1\} \). More generally, we see that a permutation which contains \( k \) specified patterns from the list 12, 23, \ldots, \((n-1)n\) can be regarded as a permutation of \( n-k \) symbols, and thus that

\[
|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!
\]

for each \( k \)-combination \( \{i_1, i_2, \ldots, i_k\} \) of \( \{1, 2, \ldots, n-1\} \). Since for each \( k = 1, 2, \ldots, n-1 \) there are \( \binom{n-1}{k} \) \( k \)-combinations of \( \{1, 2, \ldots, n-1\} \), applying the inclusion-exclusion principle we obtain the formula in the theorem. \( \square \)

Using the formula of Theorem 6.5.1, we calculate that

\[
Q_5 = 5! - \binom{4}{1}4! + \binom{4}{2}3! - \binom{4}{3}2! + \binom{4}{4}1! = 53.
\]

The numbers \( Q_1, Q_2, Q_3, \ldots \) are closely related to the derangement numbers. Indeed we have \( Q_n = D_n + D_{n-1} \), \( n \geq 2 \) (see Exercise 23). Thus knowing the derangement numbers, we can calculate the numbers \( Q_1, Q_2, Q_3, \ldots \). Since we have already seen in the preceding section that \( D_5 = 44 \), \( D_6 = 265 \), we conclude that \( Q_6 = D_6 + D_5 = 265 + 44 = 309 \).

### 6.6 Exercises

1. Find the number of integers between 1 and 10,000 inclusive which are not divisible by 4, 5, or 6.

2. Find the number of integers between 1 and 10,000 inclusive which are not divisible by 4, 6, 7, or 10.

3. Find the number of integers between 1 and 10,000 which are neither perfect squares nor perfect cubes.

4. Determine the number of 12-combinations of the multiset

\[
S = \{4 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\}.
\]
5. Determine the number of 10-combinations of the multiset
   \[ S = \{ \infty \cdot a, 4 \cdot b, 5 \cdot c, 7 \cdot d \} \].

6. A bakery sells chocolate, cinnamon, and plain doughnuts and
   at a particular time has 6 chocolate, 6 cinnamon, and 3 plain.
   If a box contains 12 doughnuts, how many different boxes of
   doughnuts are possible?

7. Determine the number of solutions of the equation \( x_1 + x_2 + x_3 + x_4 = 14 \) in non-negative integers \( x_1, x_2, x_3, \) and \( x_4 \) not exceeding 8.

8. Determine the number of solutions of the equation \( x_1 + x_2 + x_3 + x_4 = 14 \) in positive integers \( x_1, x_2, x_3, \) and \( x_4 \) not exceeding 8.

9. Determine the number of integral solutions of the equation
   \[ x_1 + x_2 + x_3 + x_4 = 20 \]
   which satisfy
   \[ 1 \leq x_1 \leq 6, \ 0 \leq x_2 \leq 7, \ 4 \leq x_3 \leq 8, \ 2 \leq x_4 \leq 6. \]

10. Let \( S \) be a multiset with \( k \) distinct objects whose repetition
    numbers are \( n_1, n_2, \ldots, n_k \), respectively. Let \( r \) be a positive
    integer such that there is at least one \( r \)-combination of \( S \). Show
    that in applying the inclusion-exclusion principle to determine
    the number of \( r \)-combinations of \( S \), one has
    \[ A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset. \]

11. Determine the number of permutations of \( \{ 1, 2, \ldots, 8 \} \) in which
    no even integer is in its natural position.

12. Determine the number of permutations of \( \{ 1, 2, \ldots, 8 \} \) in which
    exactly four integers are in their natural position.

13. Determine the number of permutations of \( \{ 1, 2, \ldots, 9 \} \) in which
    at least one odd integer is in its natural position.

14. Determine a general formula for the number of permutations
    of the set \( \{ 1, 2, \ldots, n \} \) in which exactly \( k \) integers are in their
    natural positions.

15. At a party 7 gentlemen check their hats. In how many ways
    can their hats be returned so that

    (a) no gentleman receives his own hat?
    (b) at least one of the gentlemen receives his own hat?
    (c) at least two of the gentlemen receive their own hats?

16. Use combinatorial reasoning to derive the identity
   \[ n! = \binom{n}{0} D_n + \binom{n}{1} D_{n-1} + \binom{n}{2} D_{n-2} \]
   \[ + \cdots + \binom{n}{n-1} D_1 + \binom{n}{n} D_0. \]
   (Here \( D_0 \) is defined to be 1.)

17. Determine the number of permutations of the multiset
   \[ S = \{ 3 \cdot a, 4 \cdot b, 2 \cdot c \} \]
   where, for each type of letter, the letters of the same type do
   not appear consecutively. (Thus \( abbbbcaca \) is not allowed, but
   \( abbbacabca \) is.)

18. Verify the factorial formula
   \[ n! = (n - 1)((n - 2)! + (n - 1)!), \quad (n = 2, 3, 4, \ldots). \]

19. Using the evaluation of the derangement numbers as given in
    Theorem 6.3.1, provide a proof of the relation
    \[ D_n = (n - 1)(D_{n-2} + D_{n-1}), \quad (n = 3, 4, 5, \ldots). \]

20. Starting from the formula \( D_n = nD_{n-1} + (-1)^n, \) \( (n = 2, 3, 4, \ldots), \)
    give a proof of Theorem 6.3.1.

21. Prove that \( D_n \) is an even number if and only if \( n \) is an odd
    number.

22. Show that the numbers \( Q_n \) of section 6.5 can be rewritten in
    the form
    \[ Q_n = (n-1)\left( n \frac{n-1}{1!} + \frac{n-2}{2!} - \frac{n-3}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!} \right). \]
23. (Continuation of Exercise 22.) Verify the identity
\[
(-1)^k \frac{n-k}{k!} = (-1)^k \frac{n}{k!} + (-1)^{k-1} \frac{1}{(k-1)!},
\]
and use it to prove that \( Q_n = D_n + D_{n-1} \), \( n = 2, 3, \ldots \).

24. What is the number of ways to place six non-attacking rooks on the 6-by-6 boards with forbidden positions as shown?

\[
\begin{array}{cccccc}
\times & \times & \times & & & \\
& \times & \times & \times & & \\
\times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \\
& \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times \\
\end{array}
\]

(a)

\[
\begin{array}{cccccc}
\times & \times & \times & & & \\
& \times & \times & \times & & \\
\times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \\
& \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times \\
\end{array}
\]

(b)

\[
\begin{array}{cccccc}
\times & \times & \times & & & \\
& \times & \times & \times & & \\
\times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \\
& \times & \times & \times & \times & \times & \\
\times & \times & \times & \times & \times & \times \\
\end{array}
\]

(c)

25. Count the permutations \( i_1 i_2 i_3 i_4 i_5 i_6 \) of \( \{1, 2, 3, 4, 5, 6\} \) where \( i_1 \neq 1, 5; i_3 \neq 2, 3, 5; i_4 \neq 4 \) and \( i_6 \neq 5, 6 \).

26. Count the permutations \( i_1 i_2 i_3 i_4 i_5 i_6 \) of \( \{1, 2, 3, 4, 5, 6\} \) where \( i_1 \neq 1, 2, 3; i_2 \neq 1; i_3 \neq 1, 5; i_5 \neq 5, 6 \) and \( i_6 \neq 5, 6 \).

27. Eight girls are seated around a carousel. In how many ways can they change seats so that each has a different girl in front of her?

28. Eight boys are seated around a carousel but facing inward, so that each boy faces another. In how many ways can they change seats so that each faces a different boy?

29. How many circular permutations are there of the multiset \( \{3 \cdot a, 4 \cdot b, 2 \cdot c, 1 \cdot d\} \) where for each type of letter, all letters of that type do not appear consecutively?

30. How many circular permutations are there of the multiset \( \{2 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\} \) where for each type of letter, all letters of that type do not appear consecutively?