Section 3.3 Differentiation Rules

Recall

The slope of the line tangent to graph of \( y = f(x) \) at \( (x, f(x)) \) is

\[
\lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} = \lim_{{z \to x}} \frac{f(z) - f(x)}{z - x}
\]

The derivative of a function \( y = f(x) \) is another function \( f'(x) \) defined as

\[
f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} = \lim_{{z \to x}} \frac{f(z) - f(x)}{z - x}
\]

(approximate slope of tangent to \( y = f(x) \) at \( (x, f(x)) \))

Example (from last time)

If \( f(x) = \sqrt{x} \), then

\[
f'(x) = \lim_{{h \to 0}} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{2 \sqrt{x}}
\]

\( f'(x) = \frac{1}{2 \sqrt{x}} \)

Notation

If \( y = f(x) \), then \( f'(x) = \frac{dy}{dx} \) \( \left[ f(x) \right] = \frac{df}{dx} = \frac{dy}{dx} = D_x \left[ f(x) \right] \)

If \( z = g(u) \), then \( g'(u) = \frac{dz}{du} \) \( \left[ g(u) \right] = \frac{dg}{du} = \frac{dz}{du} = D_u \left[ g(u) \right] \) etc.

Example

If \( y = f(x) \), then \( f'(x) = \frac{dy}{dx} = \frac{1}{2 \sqrt{x}} \)

\( f'(4) = \left. \frac{dy}{dx} \right|_{x=4} = \frac{1}{2 \sqrt{4}} = \frac{1}{4} \)
Section 3.3 goal:
Find formulas for derivatives that bypass limit process.

Basic Question: If \( y = f(x) \), what is \( \frac{d}{dx} [f(x)] \)?

i.e., what is the derivative?

Let's start simple. Today we will introduce 5 fundamental rules.

Rule 1 (Derivative of a constant)
If \( f(x) = c \), then \( f'(x) = 0 \)
i.e. \( \frac{d}{dx} [c] = 0 \)

Examples: \( \frac{d}{dx} [3] = 0, \frac{d}{dx} [\pi] = 0, \frac{d}{dx} [\pi^2] = 0, \frac{d}{dx} [0] = 0 \), etc.

Rule 2 (Derivative of identity)
If \( f(x) = x \), then \( f'(x) = 1 \)
i.e. \( \frac{d}{dx} [x] = 1 \)

Rule 3 (Constant multiple rule)
\[ \frac{d}{dx} [cf(x)] = c \frac{d}{dx} [f(x)] \]

Text: \( \frac{d}{dx} [cu] = c \frac{du}{dx} \)

Proof: \[ \frac{d}{dx} [cf(x)] = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c \frac{d}{dx} [f(x)] \]

Example: \( \frac{d}{dx} [5x] = 5 \frac{d}{dx} [x] = 5 \cdot 1 = 5 \)

You will get faster at this and skip steps.
If \( f(x) = 5x \), then \( f'(x) = 5 \), etc.
Rule 4  (Sum/difference rule)

\[ \frac{d}{dx} \left[ f(x) + g(x) \right] = f'(x) + g'(x) \]
\[ \frac{d}{dx} \left[ f(x) - g(x) \right] = f'(x) - g'(x) \]

Text: \( \frac{d}{dx} [u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx} \)

Text proves this via the limit definition of the derivative.
Please read and digest it.

Example \( \frac{d}{dx} [5x + 4] = \frac{d}{dx} [5x] + \frac{d}{dx} [4] \)
\[ = 5 \frac{d}{dx} [x] + 0 = 5 \cdot 1 + 0 = 5 \]

Again it’s ok to skip steps. If \( f(x) = 5x + 4 \), then \( f'(x) = 5 \)

Note: Although Rule 4 is stated for two functions, it works for any number: \( \frac{d}{dx} [f(x) + g(x) - h(x)] = f'(x) + g'(x) - h'(x) \) etc.

Rule 5  (Power Rule)

\[ \frac{d}{dx} [x^n] = nx^{n-1} \]

Let’s delay the proof of this and work some examples first.

\[ \text{Ex: } \frac{d}{dx} [x^3] = 3x^{3-1} = 3x^2 \]
\[ \text{Ex: } \frac{d}{dx} [\sqrt{x}] = \frac{d}{dx} \left[ x^{\frac{1}{2}} \right] = \frac{1}{2} \frac{x^{-\frac{1}{2}}}{x} = \frac{1}{2} \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}} \]
\[ \text{Ex: } \frac{d}{dx} \left[ \frac{1}{x} \right] = \frac{d}{dx} [x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \]

\[ \text{Ex: } \frac{d}{dx} [3x^5 + 4x - 1] = \frac{d}{dx} [3x^5] + \frac{d}{dx} [4x] - \frac{d}{dx} [1] \]
\[ = 3 \frac{d}{dx} [x^5] + 4 \frac{d}{dx} [x] - 0 \]
\[ = 3 \cdot 5x^4 + 4 \cdot 1 \]
\[ = 15x^4 + 4 \]

Notice how these match the results we got by limits earlier.

\( \text{rule 4) (rules 3, 1) (rules 5, 2) } \)

OK to do it in your head, all in one step.