Hour Exam No.1

Please attempt all of the following problems before the due date. All problems count the same even though some are more complex than others.

Problem 1

At Minkowski coordinate time, $t$, an object is located at the Minkowski position coordinates

$$x(t) = \sqrt{t^2 + 1}; \quad y(t) = z(t) = 0.$$ 

Using $c = 1$ units, find

a. the object’s ordinary Newtonian velocity vector components.

Answer 1a

$$v_x = \frac{dx}{dt} = \frac{d}{dt} \left( \sqrt{t^2 + 1} \right) = \frac{1}{\sqrt{(t^2+1)}} t$$

$$v_y = v_z = 0.$$ 

b. the components of the object’s four-velocity vector.

Answer 1b

Use the formula $u = u^0 \left( e_0 + \vec{v} \right)$ or

$$u^0 = \frac{1}{\sqrt{1 - v^2}}$$

$$u^i = u^0 v^i$$

$$v^2 = v_x^2 + v_y^2 + v_z^2 = \left( \frac{1}{\sqrt{(t^2+1)}} t \right)^2 = \frac{1}{t^2+1} t^2$$

$$1 - v^2 = 1 - \frac{1}{t^2+1} t^2 = \frac{1}{t^2+1}$$

$$u^0 = \frac{1}{\sqrt{t^2+1}} = \sqrt{(t^2 + 1)}$$

$$u^1 = u_x = u^0 v_x = \sqrt{(t^2 + 1)} \frac{1}{\sqrt{(t^2 + 1)}} t = t$$

$$u^0 = \sqrt{(t^2 + 1)}$$

$$u_x = t$$

$$u_y = u_z = 0$$
Problem 2

Consider a two-dimensional spacetime manifold where we are using the coordinates $t, x$ to locate events and the corresponding holonomic basis vectors

$$
\partial_t = \frac{\partial}{\partial t}, \quad \partial_x = \frac{\partial}{\partial x}
$$

to span each tangent space. A different coordinate system $t', x'$ also locates events in this spacetime where

$$
t' = t; \quad x' = x - \sqrt{t^2 + 1},
$$

and the corresponding holonomic basis vectors

$$
\partial_{t'} = \frac{\partial}{\partial t'}, \quad \partial_{x'} = \frac{\partial}{\partial x'}
$$

also span each tangent space.

a. Notice that $\partial_{t'}$ is not equal to $\partial_t$ even though $t' = t$. Express $\partial_{t'}$ in terms of $\partial_t$ and $\partial_x$.

Answer 2a

From the chain rule for partial derivatives,

$$
\partial_{t'} = \frac{\partial}{\partial x'} = \frac{\partial t}{\partial x'} \frac{\partial}{\partial t} + \frac{\partial x}{\partial x'} \frac{\partial}{\partial x}
$$

To evaluate these partials, we need to solve for $x, y$ in terms of $x', t'$.

$$
x' = x - \sqrt{t^2 + 1} \\
x = x' + \sqrt{t^2 + 1} = x' + \sqrt{t'^2 + 1} \\
t = t'
$$

$$
\frac{\partial t}{\partial t'} = 1, \quad \frac{\partial x}{\partial t'} = \frac{\partial}{\partial t'} (x' + \sqrt{t'^2 + 1}) = \frac{x'}{\sqrt{t'^2 + 1}}
$$

(Notice that if you use SN–Maple to do this, it gets confused by the primes.)

$$
\partial_{t'} = \frac{\partial}{\partial t} + \frac{t'}{\sqrt{t'^2 + 1}} \frac{\partial}{\partial x}
$$

b. Express $\partial_{x'}$ in terms of $\partial_t$ and $\partial_x$.

Answer 2b

$$
\partial_{x'} = \frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'} \frac{\partial}{\partial x} + \frac{\partial t}{\partial x'} \frac{\partial}{\partial t} = \frac{\partial}{\partial x}
$$
C. Quick answer: What differential forms correspond to the basis dual to the basis vectors $\partial_t$ and $\partial_x$?

Answer 2c

$$dt, dx$$

Check this (optional):

\[
\begin{align*}
\partial_t \cdot dt &= \frac{\partial t}{\partial t} = 1 \\
\partial_x \cdot dx &= \frac{\partial x}{\partial x} = 1 \\
\partial_t \cdot dx &= \frac{\partial x}{\partial t} = 0 \\
\partial_x \cdot dt &= \frac{\partial t}{\partial x} = 0
\end{align*}
\]
Problem 3

Suppose that the metric tensor on a spacetime has the form

\[ g = -dt \otimes dt + dx \otimes dx \]

and you decide to use the “null” coordinates

\[ u^1 = t + x \]
\[ u^2 = t - x \]

and the corresponding “null basis” vectors

\[ e_1 = \frac{\partial}{\partial u^1}, \quad e_2 = \frac{\partial}{\partial u^2}. \]

**a.** Express the null basis vectors in terms of the basis vectors \( \partial_t \) and \( \partial_x \) that go with the coordinates \( x, t \).

**Answer 3a**

From the chain rule for partial derivatives,

\[ e_1 = \frac{\partial}{\partial u^1} = \frac{\partial t}{\partial u^1} \frac{\partial}{\partial t} + \frac{\partial x}{\partial u^1} \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial u^2} = \frac{\partial t}{\partial u^2} \frac{\partial}{\partial t} + \frac{\partial x}{\partial u^2} \frac{\partial}{\partial x} \]

Invert the relation to get \( x, t \) in terms of \( u^1, u^2 \)

\[ t = \frac{1}{2} (u^1 + u^2), \quad x = \frac{1}{2} (u^1 - u^2) \]

\[ \frac{\partial t}{\partial u^1} = \frac{1}{2}, \quad \frac{\partial x}{\partial u^1} = \frac{1}{2}, \quad \frac{\partial t}{\partial u^2} = \frac{1}{2}, \quad \frac{\partial x}{\partial u^2} = \frac{1}{2} \]

\[ e_1 = \frac{1}{2} (\partial_t + \partial_x) \]
\[ e_2 = \frac{1}{2} (\partial_t - \partial_x) \]

**b.** Find the metric components \( g_{11}, g_{12}, g_{22} \) in the null basis.

**Answer 3b**

\[ g_{11} = \frac{1}{2} (\partial_t + \partial_x) \cdot \frac{1}{2} (\partial_t + \partial_x) = \frac{1}{4} (\partial_t \cdot \partial_t + \partial_t \cdot \partial_x + \partial_x \cdot \partial_t + \partial_x \cdot \partial_x) = \frac{1}{4} (-1 + 1) = 0 \]
\[ g_{12} = \frac{1}{2} (\partial_t + \partial_x) \cdot \frac{1}{2} (\partial_t - \partial_x) = \frac{1}{4} (\partial_t \cdot \partial_t - \partial_t \cdot \partial_x - \partial_x \cdot \partial_t + \partial_x \cdot \partial_x) = \frac{1}{4} (-1 - 1) = -\frac{1}{2} \]
\[ g_{22} = \frac{1}{2} (\partial_t - \partial_x) \cdot \frac{1}{2} (\partial_t - \partial_x) = \frac{1}{4} (\partial_t \cdot \partial_t - \partial_t \cdot \partial_x - \partial_x \cdot \partial_t + \partial_x \cdot \partial_x) = \frac{1}{4} (-1 + 1) = 0 \]
Problem 4

An electromagnetic field two-form is given by

\[ f = r^{-2}dt \wedge dr \]

where \( t \) is the usual Minkowski time function and \( r = \sqrt{x^2 + y^2 + z^2} \) is a radius coordinate. In the following, use the basis vectors

\[ \partial_0 = \frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial r} \]

and their dual basis forms and assume \( c = 1 \) units.

a. Find the components \( F_{00}, F_{01}, F_{10}, F_{11} \) of \( f \).

**Answer 4a**

\[ f = r^{-2}dt \wedge dr = r^{-2} (dt \otimes dr - dr \otimes dt) \]

The components are

\[
F_{00} = f (\partial_0, \partial_0) = r^{-2} (dt \otimes dr - dr \otimes dt) (\partial_0, \partial_0) \\
= r^{-2} (dt (\partial_0) dr (\partial_0) - dr (\partial_0) dt (\partial_0)) \\
= r^{-2} (1 \times 0 - 0 \times 1) = 0
\]

Copy the above three lines several times and just change the subscripts to get the rest.

\[
F_{01} = f (\partial_0, \partial_1) = r^{-2} (dt \otimes dr - dr \otimes dt) (\partial_0, \partial_1) \\
= r^{-2} (dt (\partial_0) dr (\partial_1) - dr (\partial_0) dt (\partial_1)) \\
= r^{-2} (1 \times 1 - 0 \times 0) = r^{-2}
\]

\[
F_{10} = f (\partial_1, \partial_0) = r^{-2} (dt \otimes dr - dr \otimes dt) (\partial_1, \partial_0) \\
= r^{-2} (dt (\partial_1) dr (\partial_0) - dr (\partial_1) dt (\partial_0)) \\
= r^{-2} (0 \times 0 - 1 \times 1) = -r^{-2}
\]

\[
F_{11} = f (\partial_1, \partial_1) = r^{-2} (dt \otimes dr - dr \otimes dt) (\partial_1, \partial_1) \\
= r^{-2} (dt (\partial_1) dr (\partial_1) - dr (\partial_1) dt (\partial_1)) \\
= r^{-2} (1 \times 1 - 1 \times 1) = 0
\]

\[
\begin{pmatrix}
F_{00} & F_{01} \\
F_{10} & F_{11}
\end{pmatrix}
= \begin{pmatrix}
0 & r^{-2} \\
-r^{-2} & 0
\end{pmatrix}
\]
b. Find the components $F^{1 \ 0}$ and $F^{0 \ 1}$ of the related tensor.

Answer 4b

$F^{1 \ 0} = F_{10} = -r^{-2}$
$F^{0 \ 1} = -F_{01} = -r^{-2}$

C. Identify the sort of object that would make this electromagnetic field.

Answer 4c

You can probably guess that this thing is a point charge. However, to get the sign and amount of the charge right, you should recall the form of the force law that we are using: For a test particle of charge $e$ and mass $m$,

$$\frac{dp}{d\tau} = \frac{e}{m}g^{-1}(f(p))$$

or, in terms of components

$$\frac{dp^\alpha}{d\tau} = \frac{e}{m}g^{\alpha \sigma}f_{\sigma \rho}p^\rho = \frac{e}{m}F^{\alpha \rho}p^\rho$$

For a test charge at rest, $p^r = 0$, $p^0 = m$, and $\tau = t$,

$$\frac{dp^1}{dt} = eF^{1 \ 0} = -er^{-2}$$

Because $e_1$ points radially, we find a radial force of $-\frac{e}{r^2}$. Compare this result to the usual form of Coulomb’s law:

$$k\frac{Qe}{r^2} = -\frac{e}{r^2}$$

The charge that generates this field must be

$$Q = -\frac{1}{k}$$

or about $-1.1 \times 10^{-10}$ Coulombs.

If you want the units to come out right, you need to attach the correct units (N/C) to the field two-form.

Problem 5 was the same as problem 3.
Problem 6

A set of observers are sitting on a flat turntable that is rotating with angular velocity $\omega$. Each observer is located at a fixed pair of Cartesian coordinates $x', y'$ that rotate with the disk. In terms of the non-rotating Minkowski coordinates $t, x, y$ the position of an given observer at $x', y'$ is

$$x = x' \cos \omega t - y' \sin \omega t$$
$$y = x' \sin \omega t + y' \cos \omega t$$

Using $c = 1$ units,

a. find the four-velocity vector $u$ of the observer at

$$x' = r$$
$$y' = 0$$

in terms of the non-rotating Minkowski basis vectors $\partial_t, \partial_x, \partial_y$

Answer 6a

First find the Newtonian velocity components of an object at constant $x', y'$. 

$$v_x = \frac{dx}{dt} = \frac{dx'}{dt} (x' \cos \omega t - y' \sin \omega t) = -x' (\sin \omega t) \omega - y' (\cos \omega t) \omega$$
$$v_y = \frac{dy}{dt} = \frac{dx}{dt} (x' \sin \omega t + y' \cos \omega t) = x' (\cos \omega t) \omega - y' (\sin \omega t) \omega$$

The value of $u^2$ we know should be $(x'^2 + y'^2) \omega^2 = r^2 \omega^2$.

Now use the formulas for the four-velocity components

$$u^0 = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-r^2 \omega^2}}$$
$$u_x = u^0 v_x = \frac{-x' (\sin \omega t) \omega - y' (\cos \omega t) \omega}{\sqrt{1-r^2 \omega^2}}$$
$$u_y = u^0 v_y = \frac{x' (\cos \omega t) \omega - y' (\sin \omega t) \omega}{\sqrt{1-r^2 \omega^2}}$$

and put these together with the basis vectors to form the four-velocity

$$u = \frac{1}{\sqrt{1-r^2 \omega^2}} (\partial_t - \omega r \sin \omega t \partial_x + \omega r \cos \omega t \partial_y)$$

b. The co-rotating coordinate system consists of $t' = t, x', y'$. Note that it is still the same $t$. Express the co-rotating holonomic basis vectors $\partial_{x'}$ and $\partial_{y'}$ at

$$x' = r$$
$$y' = 0$$

in terms of the non-rotating Minkowski basis vectors $\partial_t, \partial_x, \partial_y$
Answer 6b

First notice that the partials with respect to $x', y'$ are holding $t'$ and therefore $t$ constant and the same is true of the partials with respect to $x, y$. Thus $t$ and $t'$ are just constants at this point.

\[
\partial_{x'} = \frac{\partial}{\partial x} = \frac{\partial x}{\partial x} \frac{\partial}{\partial x} + \frac{\partial y}{\partial x} \frac{\partial}{\partial y} = \cos \omega t \partial_x + \sin \omega t \partial_y
\]

\[
\partial_{y'} = \frac{\partial}{\partial y} = \frac{\partial x}{\partial y} \frac{\partial}{\partial x} + \frac{\partial y}{\partial y} \frac{\partial}{\partial y} = -\sin \omega t \partial_x + \cos \omega t \partial_y
\]

Notice that the values of $x'$ and $y'$ do not actually matter.

C. Suppose that the observer at

\[
x' = r \\
y' = 0
\]

uses the basis vectors

\[
e_0 = u \\
e_1 = \partial_x \\
e_2 = \partial_y
\]

Find the components of the metric tensor in this basis.

Answer 6c

Collect the results of parts a, b to express the $e_i$ in terms of the Minkowski basis vectors.

\[
e_0 = \frac{1}{\sqrt{1 - r^2 c^2}} (\partial_t - \omega r \sin \omega t \partial_x + \omega r \cos \omega t \partial_y)
\]

\[
e_1 = \cos \omega t \partial_x + \sin \omega t \partial_y
\]

\[
e_2 = -\sin \omega t \partial_x + \cos \omega t \partial_y
\]

Take dot products using the usual Minkowski metric tensor.

\[
g_{00} = e_0 \cdot e_0 = u \cdot u = -1
\]

\[
g_{01} = e_0 \cdot e_1 = \frac{1}{\sqrt{1 - r^2 c^2}} (\partial_t - \omega r \sin \omega t \partial_x + \omega r \cos \omega t \partial_y) \cdot (\cos \omega t \partial_x + \sin \omega t \partial_y)
\]

\[
= \frac{1}{\sqrt{1 - r^2 c^2}} (-\omega r \sin \omega t \partial_x \cdot \cos \omega t \partial_x + \omega r \cos \omega t \partial_y \cdot \sin \omega t \partial_y)
\]

\[
= \frac{1}{\sqrt{1 - r^2 c^2}} (-\omega r \sin \omega t \cos \omega t + \omega r \cos \omega t \sin \omega t) = 0
\]

\[
g_{02} = e_0 \cdot e_2 = \frac{1}{\sqrt{1 - r^2 c^2}} (\partial_t - \omega r \sin \omega t \partial_x + \omega r \cos \omega t \partial_y) \cdot (-\sin \omega t \partial_x + \cos \omega t \partial_y)
\]

\[
= \frac{1}{\sqrt{1 - r^2 c^2}} (-\omega r \sin \omega t \partial_x \cdot (-\sin \omega t \partial_x + \cos \omega t \partial_y) + \omega r \cos \omega t \partial_y \cdot \cos \omega t \partial_y)
\]

\[
= \frac{1}{\sqrt{1 - r^2 c^2}} (\omega r \sin^2 \omega t + \omega r \cos^2 \omega t) = \frac{\omega r}{\sqrt{1 - r^2 c^2}}
\]

\[
g_{11} = e_1 \cdot e_1 = (\cos \omega t \partial_x + \sin \omega t \partial_y) \cdot (\cos \omega t \partial_x + \sin \omega t \partial_y)
\]

\[
= (\cos^2 \omega t + \sin^2 \omega t) = 1
\]

\[
g_{12} = e_1 \cdot e_2 = (\cos \omega t \partial_x + \sin \omega t \partial_y) \cdot (-\sin \omega t \partial_x + \cos \omega t \partial_y)
\]

\[
= \cos \omega t \partial_x \cdot (-\sin \omega t \partial_x) + \sin \omega t \partial_y \cdot \cos \omega t \partial_y
\]
\[ g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = (-\sin \omega t \partial_x + \cos \omega t \partial_y) \cdot (-\sin \omega t \partial_x + \cos \omega t \partial_y) \\
= (-\sin \omega t \partial_x) \cdot (-\sin \omega t \partial_x) + \cos \omega t \partial_y \cdot \cos \omega t \partial_y \\
= \sin^2 \omega t + \cos^2 \omega t = 1 \]

\[
\begin{pmatrix}
g_{00} & g_{01} & g_{02} \\
g_{10} & g_{11} & g_{12} \\
g_{20} & g_{21} & g_{22}
\end{pmatrix} = \begin{pmatrix}
-1 & 0 & \frac{\omega r}{\sqrt{1 - r^2 \omega^2}} \\
0 & 1 & 0 \\
\frac{\omega r}{\sqrt{1 - r^2 \omega^2}} & 0 & 1
\end{pmatrix}
\]

**Added Note:**
At the point \( x' = r, y' = 0 \) the basis vectors \( e_1 = \partial_{r'}, e_2 = \partial_{y'} \) can be expressed in terms of polar coordinates on the turntable as
\[ e_1 = \partial_r \]
\[ e_2 = r^{-1} \partial_\varphi \]
\[ \partial_\varphi = r e_2 \]

so that the corresponding metric components are
\[
\begin{pmatrix}
g_{00} & g_{0r} & g_{0\varphi} \\
g_{r0} & g_{rr} & g_{r\varphi} \\
g_{\varphi0} & g_{r\varphi} & g_{\varphi\varphi}
\end{pmatrix} = \begin{pmatrix}
-1 & 0 & \frac{\omega r^2}{\sqrt{1 - r^2 \omega^2}} \\
0 & 1 & 0 \\
\frac{\omega r^2}{\sqrt{1 - r^2 \omega^2}} & 0 & r^2
\end{pmatrix}
\]

and the standard form of the metric is
\[ ds^2 = -dt^2 + 2 \frac{\omega r^2}{\sqrt{1 - r^2 \omega^2}} dt d\varphi + r^2 d\varphi^2 \]

The cross-term in \( dt d\varphi \) is the signal that this is the metric of a rotating system.