Exercise 01

Please attempt all of the following problems before the due date. Your grade on this assignment will be calculated from the best two answers.

Problem 01.1

(Module 006) Consider arrows drawn on a flat sheet of paper, with addition defined in the usual graphical head-to-tail fashion, and multiplication defined to act on only the length of the vector. Use Euclidean geometry to confirm that this system is a vector space.

Answer 01.2

There are two basic approaches.

One is to use Euclidean geometry to establish that the head-to-tail method of vector addition corresponds to adding Cartesian components and multiplying the length of a vector without changing its direction multiplies its components by the same factor. The first one follows because perpendicular segments between parallel lines all have the same length. The second one comes from similar triangles having proportional length sides. Once the correspondence with $\mathbb{R}^2$ is established, you are done because that is known to be a vector space.

The other approach is to verify the vector space axioms directly using Euclidean geometry. One interesting axiom to verify is the commutation of addition, which corresponds to showing that parallel line segments between parallel lines have the same length so that the vector additions $\vec{A} + \vec{B}$ and $\vec{B} + \vec{A}$ form a closed parallelogram.

The Associative Law for vector addition can be shown by noting that the vectors $\vec{A}, \vec{B}, \vec{C}$ can be thought of as trip from point 1 to point 2, from point 2 to point 3, and from point 3 to point 4. The sum $\vec{A} + \vec{B}$ is a trip from point 1 to point 3. Since $\vec{C}$ takes you from point 3 to point 4, we have

$$(\vec{A} + \vec{B}) + \vec{C}$$

as a trip from point 1 to point 3 to point 4. Similarly,

$$\vec{A} + (\vec{B} + \vec{C})$$

is a trip from point 1 to point 2, followed by a trip from point 2 to point 4 and ends up in the same place.

The Distributive Law can be shown by considering the triangle consisting of vectors $\vec{A}, \vec{B}$ placed head-to-tail and their resultant $\vec{A} + \vec{B}$. The triangle consisting of vectors $k\vec{A}$ and $k\vec{B}$ placed head-to-tail will be a similar triangle, so the third side of the triangle will be multiplied by $k$ and one has

$$k\vec{A} + k\vec{B} = k(\vec{A} + \vec{B}).$$
The remaining vector space axioms amount to showing the correspondence between points along a line and the real numbers.

**Problem 01.2**

(Module 007) Consider the vector space $\mathbb{R}^2$ of real number pairs with addition defined by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and multiplication by a scalar defined by

$$a (x, y) = (ax, ay).$$

Use the definition of linear independence to show that the number pairs $(1, 0), (0, 1), (1, 1)$ are not linearly independent.

**Answer 01.2**

We seek non-zero coefficients $a_1, a_2, a_3$ such that

$$a_1 (1, 0) + a_2 (0, 1) + a_3 (1, 1) = 0$$

or, adding components

$$(a_1 + a_3, a_2 + a_3) = 0$$

or

$$a_1 + a_3 = 0$$
$$a_2 + a_3 = 0$$

These are two equations for three unknowns. Use the first one to eliminate $a_1$.

$$a_1 = -a_3$$

Use the second one to eliminate $a_2$

$$a_2 = -a_3$$

We can get a solution by picking $a_3$ to be anything we like. In particular $a_3 = 1$ so that $a_1 = a_2 = -1$ is a non-trivial solution. Thus, the vectors are not linearly independent.

**Problem 01.3**

(Module 007) Write the expansion of a vector $v$ in the form

$$v = \begin{bmatrix} 7 & 4 & 8 & 6 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$
and use Scientific Notebook to do the following: (1) Copy the expression to the
answer section below. Mark the matrix product with the mouse and press Ctrl-
E to expand it. (2) Construct the expression yourself by locating the "matrix"
tool and creating a 1 by 4 matrix and a 4 by 1 matrix.

Answer 01.3

The result of the evaluation is

$$7e_1 + 4e_2 + 8e_3 + 6e_4$$

To construct the expression, go to the "view\|toolbars" menu and make sure
that the box next to "Math Objects" is checked. Use the button with a picture
of an array on it. That should bring up a dialogue box titled "Matrix". Choose
4 columns and 4 row and click "OK". An array of boxes should appear. If you
just put numbers in the boxes, it looks like this:

$$\begin{array}{cccc}
7 & 4 & 8 & 6 \\
\end{array}$$

It is a good idea to mark the entire array with your mouse and press CTRL-(
so that the array looks like this:

$$\begin{pmatrix}
7 & 4 & 8 & 6 \\
\end{pmatrix}.$$  

The other array is created in the same way, with 1 column and 4 rows.

$$\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
\end{pmatrix}$$

Creating both arrays next to each other yields

$$\begin{pmatrix}
7 & 4 & 8 & 6 \\
\end{pmatrix} \begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
\end{pmatrix}$$