1 Problems with the Schwarzschild Metric

The spacetime outside of a non-rotating star (or planet or whatever) of total mass $m$ is described by the metric tensor

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2d\Omega^2$$

This metric solves the vacuum Einstein equations and, according to Birkhoff’s Theorem, is the only spherically symmetric metric that does. It obviously has a problem at $r = 2m$. One metric coefficient blows up while another goes to zero.

Consider a clock that is holding its position at a constant value of the radial coordinate. If it does this for an interval $\Delta t$ of coordinate time, then the time elapsed on the clock will be

$$\Delta \tau = \Delta t \sqrt{1 - \frac{2m}{r}}$$

For $r$ near $2m$ very little proper time will elapse on the clock even though a great deal of coordinate time elapses. This result tells us that the $t =$ constant hypersurfaces are failing to advance in time near $r = 2m$.

During the early days of general relativity, the problem with the Schwarzschild solution was regarded mostly as a curiosity of little consequence because, for normal astronomical objects such as the sun and the earth, the critical value of the radius is extremely small. For an object with the mass of the earth, the critical radius is about a centimeter. For the sun, it is a kilometer. The metric inside an ordinary star is not given by the Schwarzschild vacuum solution and is quite regular everywhere. For the external vacuum spacetime to extend to the critical radius, the sun would have to be compressed to a radius of a kilometer. Until 1939, (Oppenheimer and Volkoff, Oppenheimer and Snyder) no astrophysicist seriously believed that a mass equal to that of the sun could be compressed into an object only a kilometer in radius. Not too many believed it after that either.

From a mathematical point of view, the coordinate-independent nature of spacetime geometry was not well understood during the early days, so it was some time before someone considered the possibility that the $r = 2m$ singularity was simply the result of bad coordinates. One indication that this might be the case is that none of the scalar curvature invariants such as $R$, $R^\sigma_{\mu\nu\rho}$, and invariant ratios of lightlike components of curvatures become infinite at $r = 2m$.

2 The Kruskal Extension

2.1 Orthogonal Surface Metric

Once it is realized that the $r = 2m$ singularity is really just a problem with the coordinates, it is fairly easy to fix. The angle coordinates $\theta, \varphi$ are fixed by the
spherical symmetry group, so nothing could be wrong with them. That leaves
the coordinates \( r, t \) and the metric

\[ 2ds^2 = -fdt^2 + f^{-1}dr^2, \quad f = 1 - \frac{2m}{r} \]
on the timelike two-surfaces orthogonal to the group orbits. We need to find
new coordinates in which this metric looks more regular.

An obvious geometrical feature of this two dimensional metric is the light
cone. On any such surface, it is always possible to find coordinates \( U, V \) such
that curves at constant \( U \) are lightlike and so are curves at constant \( V \). For
example, in two dimensional Minkowski spacetime, one can take \( U = t + r, V = t - r \)
and obtain the metric in the form \( 2ds^2 = -dUdV \).

In general, so long as the metric takes the form

\[ 2ds^2 = -\Phi dUdV \]
then an interval with either \( \Delta U = 0 \) or \( \Delta V = 0 \) will be lightlike. Thus, we seek
coordinates \( U, V \) such that

\[-fdt^2 + f^{-1}dr^2 = -\Phi dUdV \]
for some function \( \Phi \). Once we find such coordinates, we can construct space and
time coordinates \( u = U - V, v = U + V \) in which the light cones look exactly
like the ones in Minkowski spacetime. If there is any coordinate system in which
this metric tensor is regular, then this has to be the one.

2.2 Conditions on Advanced and Retarded Time Coordinates

Represent partial derivatives with respect to \( t, r \) by subscripts so that

\[ dU = U_r dr + U_t dt, \quad dV = V_r dr + V_t dt \]
and therefore

\[-fdt^2 + f^{-1}dr^2 = -\Phi (U_r dr + U_t dt) (V_r dr + V_t dt) \]
or
\[
\Phi U_t V_t = f, \quad \Phi U_r V_r = -f^{-1}
\]
\[U_r V_t + U_t V_r = 0\]
Divide all of these equations by \(U_r V_r\) and obtain the results
\[
\frac{U_t}{U_r} \frac{V_t}{V_r} = -f^2, \quad \frac{V_t}{V_r} + \frac{U_t}{U_r} = 0,
\]
which can be solved for the two ratios in the form
\[
\frac{U_t}{U_r} = f, \quad \frac{V_t}{V_r} = -f.
\]
Thus, if the coordinates exist, then they must satisfy these two conditions. Conversely, the argument can be reversed to show that solving these two conditions is enough to put the spacetime metric into the desired form where the conformal factor \(\Phi\) can be found from the equation \(\Phi U_t V_t = f\).

In more explicit form the conditions to be solved are
\[
\frac{\partial U}{\partial t} = \left(1 - \frac{2m}{r}\right) \frac{\partial U}{\partial r}, \quad \frac{\partial V}{\partial t} = -\left(1 - \frac{2m}{r}\right) \frac{\partial V}{\partial r}
\]

### 2.3 Solving the Conditions: Tortoise Coordinate

To see what to do next, recall what these conditions would look like for null coordinates in Minkowski spacetime:
\[
\frac{\partial U}{\partial t} = \frac{\partial U}{\partial r}, \quad \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial r},
\]
which would yield the result that \(U\) depends only on \(t + r\) and \(V\) depends only on \(t - r\). To get the conditions into this form, we need a new radial coordinate \(r^*\) such that
\[
\left(1 - \frac{2m}{r}\right) \frac{\partial}{\partial r} = \frac{\partial}{\partial r^*}
\]
From the chain rule for partial derivatives,
\[
\frac{\partial}{\partial r} = \frac{\partial r^*}{\partial r} \frac{\partial}{\partial r^*}
\]
so we have
\[
\left(1 - \frac{2m}{r}\right) \frac{\partial r^*}{\partial r} \frac{\partial}{\partial r^*} = \frac{\partial}{\partial r^*}
\]
or
\[
\frac{\partial r^*}{\partial r} = \frac{1}{1 - \frac{2m}{r}}
\]
which can be re-written as follows
\[
\frac{\partial r^*}{\partial r} = \frac{r}{r - 2m} = 1 + \frac{2m}{r - 2m}
\]
and integrated directly:

\[ r^* = r + 2m \ln |r - 2m| \]

This coordinate has been called the “tortoise coordinate” after Zeno’s paradox and the way that the coordinate creeps up on the value \( r = 2m \), which is achieved only as \( r^* \to -\infty \). For large values of \( r \), the new radius coordinate becomes quite similar to \( r \).

In terms of the tortoise coordinate, we have the conditions

\[ \frac{\partial U}{\partial t} = \frac{\partial U}{\partial r^*}, \quad \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial r^*}, \]

which simply mean

\[ U = U (t + r^*), \quad V = V (t - r^*). \]

We still have two functions, \( U (x) \), \( V (x) \) of a single variable to choose.

One potential difficulty needs to be anticipated at this point. The “tortoise coordinate” \( r^* \) is not continuous across the boundary at \( r = 2m \). When we get the final form of the metric, we need to be sure that it is regular there.

### 2.4 Regularizing the Metric

Now all we have to do is find the conformal factor \( \Phi \) and see if there is any way to choose the functions \( U, V \) so that the singularity at \( r = 2m \) goes away. Use primes to denote derivatives of functions of a single variable and write the condition that is to be solved for \( \Phi \)

\[ \Phi U_t V_t = f \]

\[ U_t = \frac{\partial}{\partial t} U (t + r^*) = U' (t + r^*) \]

\[ \Phi U' V' = 1 - \frac{2m}{r} = \frac{r - 2m}{r}. \]

For this to work, the conformal factor must not vanish at \( r = 2m \) or we will still have a singularity there. Thus, the functions \( U, V \) must be chosen so that they produce a factor of \( r - 2m \). This can be arranged because

\[ U = U (t + r^*) \]

\[ = U (t + r + 2m \ln |r - 2m|) \]

and

\[ V = V (t - r^*) \]

\[ = V (t - r - 2m \ln |r - 2m|) \]

depend explicitly on \( r - 2m \). To pull out the over-all factor of \( r - 2m \) that we need, choose the functions to be exponentials

\[ U (x) = Ae^{\pm \pi r}, \quad V (x) = Be^{-\pi r}. \]
The constants $A$ and $B$ will be chosen later. The condition to be solved for the conformal factor $\Phi$ is then

$$-\Phi AB \frac{1}{4m} e^{\frac{t+r}{4m}} \frac{1}{4m} e^{-\frac{t-r}{4m}} = \frac{r - 2m}{r}$$

or

$$-\frac{AB}{16m^2} e^{\frac{t+r}{4m}} = \frac{r - 2m}{r}$$

or

$$-\frac{AB}{16m^2} e^{\frac{t+2m \ln|r-2m|}{2m}} = \frac{r - 2m}{r}$$

or

$$-\frac{AB}{16m^2} e^{\frac{t}{2m} + \ln|r-2m|} = \frac{r - 2m}{r}$$

or

$$-\Phi AB \frac{|r - 2m|}{r - 2m} = \frac{16m^2}{r} e^{-\frac{r}{2m}}.$$}

Now we see why the constants $A$ and $B$ have been carried along. They can each be either $+1$ or $-1$ but their definitions must switch as the $r = 2m$ boundary is crossed so that

$$-AB |r - 2m| = r - 2m,$$

which yields the final result

$$\Phi = \frac{16m^2}{r} e^{-\frac{r}{2m}}$$

and puts the orbit-orthogonal 2-metric into the form

$$2ds^2 = -\frac{16m^2}{r} e^{-\frac{r}{2m}} dUdV,$$

which no longer has any trace of irregularity at $r = 2m$. It remains singular at $r = 0$, however.

### 2.5 Understanding Kruskal’s Coordinates

The lightlike Kruskal coordinates $U$ and $V$ are related to the usual Schwarzschild coordinates by

$$U = Ae^{\frac{1}{2m}(t+r)}, \quad V = Be^{-\frac{1}{2m}(t-r)}$$

so that

$$UV = AB e^{\frac{r}{2m}} = AB e^{\frac{1}{2m}(r+2m \ln|r-2m|)} = AB e^{\frac{r}{2m}} |r - 2m|$$

or

$$UV = -e^{\frac{r}{2m}} (r - 2m)$$

and

$$\frac{V}{U} = \frac{B}{A} e^{\frac{r}{2m}}$$
All of these relationships have been derived for the region of spacetime where the Schwarzschild coordinates make sense. Within this region we have $U > 0$, $V < 0$ with the $r = 2m$ singularity happening at $U = V = 0$. For the definitions of $U$ and $V$ to work in this region, we need $A = 1, B = -1$. Since $r > 2m$ in this region, we confirm that the relation $\frac{-AB}{r-2m} = r - 2m$ works there.

Now switch to the space and time coordinates defined by

$$U = v + u, \quad V = v - u$$

for which the spacetime metric takes the form

$$2ds^2 = 16m^2 e^{\frac{r}{2m}} (-dv^2 + du^2).$$

In these coordinates the surfaces of constant $r$ are given by

$$v^2 - u^2 = -e^{\frac{r}{2m}} (r - 2m),$$

which implicitly defines $r(u, v)$. Similarly, the constant $t$ surfaces are given by

$$\frac{v - u}{v + u} = A B e^{\frac{t}{2m}}.$$

The region, within which we have derived the new coordinate system, corresponds to a wedge of the full $u, v$ plane:

$$v > u, \quad v > -u$$

or equivalently

$$u > |v|.$$

The boundary of this region at $u = |v|$ corresponds to $r = 2m$. Within this region, surfaces of constant $r > 2m$ correspond to hyperbolas. Surfaces of constant time correspond to straight lines through the origin with $t = -\infty$ at
$u = -v$ and $t = +\infty$ at $u = +v$.

The nature of the coordinate problem at $r = 2m$ can now be seen very clearly. The Schwarzchild time coordinate $t$ fails to advance at that point.

### 2.6 Extending the Solution

Now consider the other quadrants of the Kruskal plane. In the upper quadrant, $U > 0, V > 0$ so that the constants $A$ and $B$ can each be chosen to be $+1$. The sign of $r - 2m$ is negative, so we confirm that $-AB|r - 2m| = r - 2m$ in this region as required. Similarly, the required signs work out in each of the other quadrants. Since nothing goes wrong with the spacetime metric at the $r = 2m$ boundaries of the coordinates that we are using, we can simply use the whole solution, for all values of $u, v$. The only place where something still goes wrong is at $r = 0$ where the timelike 2-surface metric is still regular, but the group orbits have zero area.

A map of the full solution in Kruskal coordinates is called the Kruskal diagram. The region between the two hyperbolas is the maximal extension of the Schwarzschild metric. The Schwarzschild metric itself corresponds to the right-hand quadrant of the map.

To use this diagram, note that light rays move along $45^\circ$ lines so that the allowed world-lines of particles always move upward at less than $45^\circ$ to the vertical.
The diagonal lines in the picture correspond to $r = 2m$. Light signals from the right-hand quadrant of the picture can reach the outside universe. However, if an object’s world-line carries it across the $r = 2m$ surface, then any light signals that it might emit will hit the singularity and never get out.

It is important to remember that the Kruskal solution is the maximal extension of the vacuum solution. Most physical situations will use only part of it as an exterior solution, which is matched to an interior solution with matter sources. For a normal stable object such as the Earth or the Sun, only a part of the right-hand quadrant is used and the $r = 2m$ surface never appears. A collapsing star uses more of the solution because its surface passes through $r = 2m$ and uncovers a bit of the upper quadrant.

Even in that extreme case, however, the bottom quadrant and the left-hand quadrant do not appear.

There is no known situation that requires the use of the bottom quadrant of the Kruskal extension. Such a situation would be extremely awkward because signals could travel from the initial singularity to the outside universe. Because no initial conditions can be defined at the singularity, anything at all could come out of it and the entire universe would become completely unpredictable. The final singularity in the Kruskal extension does not pose such a problem because it is cloaked by the $r = 2m$ event horizon. A singularity that is not cloaked by a horizon in this way is called a naked singularity. The Cosmic Censorship Conjecture asserts that nature always manages to avoid creating naked singularities.
3 Black Holes

3.1 A little astrophysics

Stars spend most of their lives fusing hydrogen to make helium. In smaller stars such as our Sun, the fusion reaction takes place only at the center of the star. When the hydrogen fuel runs out at the center, the nuclear fire burns outward and causes the surface of the star to swell up by perhaps a factor of a hundred. The star then becomes a red giant. For our sun, the hydrogen fuel will eventually run out and the red-giant stage will be followed by a cooling white dwarf star, held up by the fact that electrons obey an exclusion principle that causes them to resist being forced into the same states. Usually the collapse to a white dwarf is accompanied by the release of a last burst of energy that blows away part of the star’s atmosphere as an expanding “planetary nebula” such as the Helix nebula below.

The newborn white dwarf star can be seen at the center of the expanding cloud. Hundreds of newborn white dwarfs are seen in the sky. Because the expanding cloud dissipates in just a few thousand years, there are many more older white dwarf stars that have lost their nebulae.

Stars much more massive than our sun go through a more elaborate series of stages but still end up in a final collapse. Often, they have too much mass to be stable white dwarfs and collapse until they are just a few kilometers in diameter. The electrons no longer support the star because they no longer exist. All of the electrons have been absorbed into neutrons through inverse beta decay and it is the exclusion principle for neutrons that holds up the star. The gravitational potential energy released when the star collapses to such a small size is comparable to all of the nuclear energy released by the star during its entire lifetime. The sudden release of such a large amount of energy causes much of the star to be blown away in a supernova explosion. Here is an image
of a supernova that occurred in 1987:

Eventually, all that is left of the star is an expanding cloud of gas with a rapidly rotating neutron star at its core. In some cases the neutron star drags its magnetic field through the nearby nebula, accelerating particles and generating enough energy to light up the whole nebula. The Crab nebula is an example of such a system.

There are some truly massive stars with 50 to 100 times the mass of our sun. Such super-massive stars are very unstable, as the case of eta carinae shown
when stars of this sort finally run out of fireworks, the mass left behind can be too much for even a neutron star and the surface area of the star drops below $4\pi (2m)^2$. The surface of the star is then inside the $r = 2m$ limit of the Kruskal metric and all timelike curves lead toward smaller values of $r$ and thus smaller values for the surface area. The star is then doomed to final collapse, no matter what physical processes go on inside it.

3.2 Are black holes really out there?

Briefly: Yes. An isolated black hole would be impossible to find since it is only a few miles across and, of course, it is black because it absorbs all light that hits it. However, matter that falls into a black hole releases enormous amounts of energy. For example, matter falling into a solar-mass size hole would fall into a gravitational field identical to that of the sun, a distance of a half million miles closer to the ‘center’ of the field than the surface of the sun, with the gravitational field increasing the whole way. Figure out the amount of work a given mass $m$ could do if it is lowered gradually into a black hole at the end of a string. The answer turns out to be $mc^2$. A black hole has the potential to convert mass into energy with 100% efficiency. It is generally believed that the most violent processes in the universe are powered by matter accreting onto black holes.

When a black hole is consuming matter, the matter usually starts out with some angular momentum so, instead of going straight into the hole, it forms
an accretion disk around the hole. In such a disk, the faster inner ring usually couples to the slower outer ring (often through embedded magnetic fields) and transfers the angular momentum outward until it is lost in a radial “plume” of ejected matter. The gravitational field of the black hole is not Newtonian and, for matter that comes too close, there are no longer any stable orbits. As the inner ring of matter is slowed, it dives into the hole. The energy released by that matter’s descent into the hole’s gravitational field is released as X-rays. Meanwhile, the twisting magnetic field generated by the rotating accretion disk causes some matter to escape along the rotation axis of the system and be focused into “jets”. Such a jet can be seen in some detail in the pictures of the galaxy M87 below:

The black hole responsible for jets such as the one seen in M87 are thought to have masses that are millions of times the mass of our sun. It has long been believed that a giant black hole of this sort exists at the centers of many galaxies, including our own.

4 Wormholes

4.1 The Einstein-Rosen Bridge

Suppose that we somehow have a spacetime that is the full Kruskal extension of the Schwartzchild vacuum solution. Consider the space geometry at the instant of time symmetry, defined in Kruskal coordinates by $v = 0$. Specializing the
implicit definition of the Schwartzchild luminosity radius coordinate $r$ to this case gives

$$u^2 = e^{-\frac{r}{2m}} (r - 2m).$$

For the dimensionless coordinates

$$k = \frac{u}{\sqrt{2m}}, \quad s = \frac{r}{2m}$$

we have the relation

$$k^2 = e^{\frac{r}{2m}} (s - 1),$$

which, when plotted, shows that the luminosity radius goes through a minimum value of $r = 2m$ at $k = 0$ or $u = 0$.

The luminosity radius goes to infinity, corresponding to being far from the source of the gravitational field when $u$ increases to infinity. It does exactly the same thing when $u$ decreases to minus infinity. Thus, we have two ‘outside’ regions.

Recall that the luminosity coordinate $r$ is defined in terms of the area of the spherical group orbits. Thus, the two outside regions are connected by a sphere of minimum area, called the ‘wormhole throat’. To make a picture of this structure, take the dimensionless luminosity radius $s$ to be a radius given by $s = \sqrt{x^2 + y^2}$ and plot the resulting circles as a function of the reduced Kruskal coordinate, $k$. The resulting surface is defined by

$$k = \sqrt{e^{\frac{\sqrt{x^2 + y^2}}{\sqrt{s^2 + y^2} - 1}}.}$$

Turn this into a parameterized plot. For this to work, you need to first define a function $K$ by

$$K (r) = \begin{cases} \sqrt{e^{\frac{r}{2m}} (r - 1)} & r > 1 \\ 0 & r \leq 1 \end{cases}$$

and then form the coordinate vector

$$[r \cos \theta, r \sin \theta, K (r)]$$
Select Compute|Plot 3D|rectangular with the cursor anywhere in this vector. Click on the resulting picture and then on the box in the lower right corner in order to adjust the parameter ranges set to $r > 0$, and $0 \leq \theta < 2\pi$. The resulting plot is

![Wormhole Throat](image)

If you are viewing this text with Scientific Notebook or Scientific Workplace, you can use your mouse to rotate the picture and view it from different angles (double click and drag).

### 4.2 Wormholes??

An Einstein-Rosen Bridge can be thought of as connecting two different universes, each of which looks like flat space-time far from the bridge. Imagine cutting the bridge and looking at each of these two universes. To make a picture, define the function

$$q(r) = \begin{cases} \frac{1}{r} & r > 0.5 \\ -16 & r \leq 0.5 \end{cases}$$
and do a 3D plot of the surface twice $[r \cos \theta, r \sin \theta, q(r)]$

Because these two universes are so close to flat spacetime at great distances, we only have to bend them a little to fit them together and have them both be the same universe with two widely separated holes like this:

$$[r \cos \theta, r \sin \theta, q \left( \sqrt{(r \cos \theta - 3)^2 + r^2 \sin^2 \theta} \right) + q \left( \sqrt{(r \cos \theta + 3)^2 + r^2 \sin^2 \theta} \right)$$

Now we have two more loose edges to take care of, namely the two throats. They fitted together before, so they will surely still match up now. Identify the two throats so that anything going down into one comes out the other. The result is an argument that Einstein’s field equations should permit a spacetime with a kind of built-in “subway” that lets you go from one part of the universe to another without traversing the ordinary space and time between.
4.3 Interstellar superhighways?

Briefly: No. Near the wormhole throat, the spacetime geometry should be almost exactly the same as for an Einstein-Rosen Bridge and will be described by the Kruskal extended vacuum solution. For two ships passing through the wormhole, we would draw a diagram like this:

![Diagram of wormholes](image)

The ship that makes it through without hitting the singularity and getting crushed to infinite density is moving on a faster-than-light trajectory. The other ship has obviously suffered a failure of its ftl drive system and is in big trouble. A faster-than-light drive is even more problematic than wormholes, so these things might not be all that useful. A worse problem can also be seen in the diagram: The initial singularity is uncovered (a ‘naked’ singularity) and can send completely unpredictable things into the region of the throat. If the initial singularity is covered by the surface of a collapsing star, then that star will block the throat.

More complex sorts of wormholes are possible. For example, the gravitational field of a rotating, electrically charged star can be extended in much the same way as the field of an uncharged, non-rotating star. In that case, the singularities may be avoidable but naked singularities are still present and there is no known way to form these things.

4.4 Then what good are they?

The fact that Einstein’s theory admits solutions like the wormhole spacetimes indicates that the underlying topology of spacetime might not be simple. It also raises the question of how that topology is determined. In Einstein’s theory one proceeds by assuming an underlying manifold with a given topology. The wormhole solutions and many other examples suggest that Einstein’s field equations can be solved for any underlying manifold. The question then is “What determines the underlying manifold?”

That question has become far more serious now that an ever-expanding variety of underlying manifold structures is being tried in particle and quantum-gravity theories, from string theory (2 dimensions) through M-theory (with an 11 dimensional low energy limit). Some theories have extra dimensions that are
rolled up into closed manifolds of several sorts while other theories have extra
dimensions that are not limited at all. The picture on this course’s home page
depicts strings that end on 4-dimensional manifolds called “branes”. How does
one find the underlying topological structure? The only answer that we have
so far is very unsatisfactory: One guesses an underlying structure and builds a
theory on it.