Measure of a fuzzy set. The $\alpha$-cut approach in the finite case

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Abstract

In this paper we study the problem of constructing a Sugeno measure $\tilde{m}$ on a suitable family $\tilde{\mathcal{F}}$ of fuzzy subsets of a space $\Omega$. More precisely, we suppose that there exists a measure $m$ on a crisp measurable space $(\Omega, \mathcal{F})$, we request that all the $\alpha$-cuts of the elements of $\tilde{\mathcal{F}}$ belong to the algebra $\mathcal{F}$ of the measurable crisp subsets and we suppose that the membership function assumes only finitely many values $\alpha_1, \ldots, \alpha_n$. Under these hypotheses we construct the measure $m(A)$ of the fuzzy set $A$ in terms of the measure $m(A_i)$ of its $\alpha$-cuts $A_i$, $i = 1, \ldots, n$, in the case where both $\tilde{m}$ and $m$ are decomposable. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and the problem

In previous works we studied the possibility of constructing the measure of a fuzzy subset of a space $\Omega$ with finitely valued membership functions, in terms of the $\alpha$-levels, that is in terms of the crisp subsets $\tilde{A}_i = \{ \omega | \tilde{A}(\omega) = \alpha_i \}$ (see e.g. [1]). Now, we attempt to reach the same objective starting from the $\alpha$-cuts (the crisp subsets $\tilde{A}_i = \{ \omega | \tilde{A}(\omega) \geq \alpha_i \}$). In order to motivate this research we can observe, first of all, that this last notion is more natural in a fuzzy environment than the notion of $\alpha$-levels. This is evident, in particular, when we examine the fuzzy sets while having in mind, as in the present case, the construction of a measure.

If the membership function $\tilde{A}$ assumes only a finite number of values, namely the image set is $\text{ran}(\tilde{A}) = \{ \alpha_1, \ldots, \alpha_n \}$ (finite case), then both $\alpha$-levels and $\alpha$-cuts usually have a positive measure. But let us examine what happens when the image of $\tilde{A}$ is the whole interval $[0, 1]$ (continuous case).

In the most natural measure space often the $\alpha$-levels are reduced to sets of null measure. This causes some problems when we try to extend the results obtained in the finite case to the continuous one, at least when the considered measures are not Archimedean.

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In contrast, the \( x \)-cuts maintain non-null measure even in the continuous case. This leads us to hope that the results which we obtain here (finite case) could be extended to the continuous case, even when the measure composes of an arbitrary continuous \( t \)-conorm.

A different approach to this problem has been proposed by Grabisch et al. [2]. They generalized the previous works of Murofushi and Sugeno [3,4] providing a definition of fuzzy measure based on Sugeno and Choquet integral.

**Definition 1.** A Sugeno measure \( \mu \) on a measurable space \(( \Omega, \mathcal{A} )\) is a map \( \mu : \mathcal{A} \to [0,1] \) with the following properties:

\[
\begin{align*}
\mu(\Omega) &= 1, \\
\mu(\emptyset) &= 0, \\
A \subset B &\Rightarrow \mu(A) \leq \mu(B).
\end{align*}
\]

It is decomposable (see e.g. [5, pp. 27, 192; 1]) if there exists a function \( \Phi : [0,1]^2 \to [0,1] \) (composition law) such that

\[
R, S \in \mathcal{A}, \quad R \cap S = \emptyset \Rightarrow \mu(R \cup S) = \Phi[\mu(R), \mu(S)].
\]

**Theorem 1.** It has been proved that \( \Phi \) is always a \( t \)-conorm and under the hypothesis of continuity it may be written in the following form:

\[
\Phi(x,y) = \begin{cases} 
\phi_i^{-1}[\phi_i(x) + \phi_i(y)] & \text{if } (x,y) \in ]a_i,b_i[^2, \\
\max(x,y) & \text{otherwise},
\end{cases}
\]

where \( \{]a_i,b_i[^i \in I \} \) is a finite or countable family of open disjoint intervals, \( \phi_i : ]a_i,b_i[ \to \mathbb{R} \) (local additive generator) is a strictly increasing function with \( \phi_i(a_i) = 0 \) and \( \phi_i^{-1} \) is the pseudo-inverse of \( \phi_i \) defined by \( \phi_i^{-1}(t) = \phi_i^{-1}[\min\{t,\phi_i(b_i)\}] \).

The set \( \Lambda = [0,1]\bigcup_i ]a_i,b_i[ = \{x \in [0,1] | \Phi(x,x) = x\} \) (the set of the idempotent elements) will assume great importance in the development of the present paper. It is important to remember that

\[
\text{if } x \in \Lambda \text{ then } \forall y \in [0,1] \quad \Phi(x,y) = \max(x,y).
\]

**The Problem.** We suppose that a decomposable Sugeno measure \( m \) is defined on a measurable space \(( \Omega, \mathcal{A} )\). Let \( L = \{x_0 = 0, x_1, \ldots, x_n = 1 | x_i < x_{i+1}\} \) be a finite family of values in \([0,1]\) such that \( x \in L \Rightarrow 1 - x \in L \) and let \( \tilde{\mathcal{A}} \) be the family of the \( L \)-fuzzy subsets of \( \Omega \) which have all the \( x \)-cuts in \( \mathcal{A} \).

We will determine a decomposable measure \( \tilde{m} \) defined on \( \tilde{\mathcal{A}} \) which is compatible with the measure \( m \). The decomposability of measures of a fuzzy set has the same meaning as that of the classical one, that is \( \tilde{A} \cap \tilde{B} = \emptyset \Rightarrow \tilde{m}(\tilde{A} \cup \tilde{B}) = \tilde{\Phi}[\tilde{m}(\tilde{A}),\tilde{m}(\tilde{B})] \), whereas the compatibility condition means that \( \tilde{m}(\tilde{A}) \) is a suitable function of the measures \( m(A_i) \) of the \( x \)-cuts \( A_i = \{\omega \in \Omega | A(\omega) \geq x_i\} \) of \( \tilde{A} \)

\[
\tilde{m}(\tilde{A}) = G[m(A_1), \ldots, m(A_n)].
\]

The second compatibility condition would be

\[
\tilde{A} = A \text{ is crisp } \Rightarrow \tilde{m}(\tilde{A}) = m(A)
\]

(complete coherence condition). Although this condition seems to be quite natural, cases may exist where it does not hold. We justify this assessment by observing that the outcome of the crisp event \( A \) has different
meanings in the context of algebra $\mathcal{S}$ (where all the events are crisp) and in the context of family $\mathcal{S}'$ (which contains fuzzy events beyond the crisp ones).

Our problem consists of determining the most general form of function $G$ satisfying condition (3) and possibly (4). The solution which we give here is restricted to the case where the measures $\tilde{m}$, $m$, which are decomposable by hypothesis, also have continuous corresponding composition laws $\tilde{F}$, $F$. So they are characterized by the result of Theorem 1. We will denote by $\tilde{A}, A$ the sets of the idempotents, $[\tilde{a}_i, \tilde{b}_i], [a_i, b_i]$ the open intervals appearing in (2), and by $\tilde{f}_i, f_j$ the local additive generators of the composition laws $\tilde{F}, F$, respectively.

We conclude this paragraph by establishing a fundamental functional equation which follows from the fact that the measure $\tilde{m}(\tilde{A} \cup \tilde{B})$ may be obtained in two different ways: (1) first compute $\tilde{m}(\tilde{A}), \tilde{m}(\tilde{B})$ by means of the function $\tilde{G}$, then compose these measures by $\tilde{F}$; (2) first compose $m(A_i), m(B_i)$ by means of the function $F$, then compose these by the function $\tilde{G}$. By setting $x_i = m(A_i)$, $y_i = m(B_i)$, the fundamental compatibility functional equation which we obtain is

$$ \tilde{F}[\tilde{G}(x_1, \ldots, x_n), \tilde{G}(y_1, \ldots, y_n)] = \tilde{G}[F(x_1, y_1), \ldots, F(x_n, y_n)]. $$

(5)

Moreover it is easy to recognize that, for $G$ to define a measure, its domain must be the set $\Gamma = \{(x_1, \ldots, x_n) | x_i \geq x_{i+1}\}$, and moreover

\begin{align*}
G(0, \ldots, 0) &= 0, \quad \text{(6)} \\
G(1, \ldots, 1) &= 1, \quad \text{(7)} \\
G &\text{ is non-decreasing.} \quad \text{(8)}
\end{align*}

**Definition 2.** $G$ is a fuzzy measure function (FMF) if it is continuous and satisfies conditions (5)–(8).

**Definition 3.** An FMF $G$ is completely coherent if the following further condition holds:

$$ G(x, \ldots, x) = x, \quad \text{(9)} $$

which corresponds to complete coherence condition (4).

2. The fuzzy measure of a crisp set

In this section we will determine the form of the fuzzy measure of a crisp subset, that is of a subset with membership function

$$ \tilde{A}(\omega) = A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(10)} $$

Let us remember that all the $\alpha$-cuts of such a set are equal to the crisp subset $A$. So, if we introduce the function

$$ g(x) = G(x, \ldots, x), \quad \text{(11)} $$

we have

$$ \tilde{m}(A) = g[m(A)]. \quad \text{(12)} $$
It is easy to recognize that the fundamental equation (5) and the properties (6)–(8) determine the following conditions which characterize the function \( g \):
\[
\tilde{F}[g(x), g(y)] = g[F(x, y)],
\]  
(13)
g is continuous,
g is non-decreasing,
g(0) = 0, \quad g(1) = 1.
(14)

**Proposition 1.** \( g(\Delta) = \tilde{\Delta} \).

**Proof.** From Eq. (13), also remembering that, if \( x \in \Delta \), then \( F(x, x) = x \), we have
\[
\tilde{F}[g(x), g(x)] = g(x).
\]
This proves that \( g(x) \in \tilde{\Delta} \), that is \( g(\Delta) \subseteq \tilde{\Delta} \). To prove that \( \tilde{\Delta} \subseteq g(\Delta) \) let us suppose that there exists \( z \in \tilde{\Delta} \), \( z \notin g(\Delta) \). Since \( g \) is surjective, there exists \( x \in [a, b] \subseteq \Delta \) such that \( z = g(x) \). Letting \( y = x \) in Eq. (13), and remembering that \( z \in \Delta \), but \( x \notin \Delta \), we have \( z = g(x) = \tilde{F}[g(x), g(x)] = g[F(x, x)] = g[\tilde{F}_i^{-1}(2f_i(x))] \) and by recursivity
\[
z = g[\tilde{F}_i^{-1}(nf_i(x))].
\]
Since \( f_i(x) > 0 \) we will have: \( z = \lim_{n \to +\infty} g[\tilde{F}_i^{-1}(nf_i(x))] = g(b_i) \), which contradicts our hypothesis. \( \square \)

**Remark.** As an immediate consequence of this proposition we can conclude that, if \( I = \text{card}(\{a_i, b_i\}) < +\infty \), then \( g[a_i, b_i[ = ]\tilde{a}_i, \tilde{b}_i[ \), whereas if \( I = +\infty \), then either \( g[a_i, b_i[ = ]\tilde{a}_j, \tilde{b}_j[ \) for some \( j \), or \( g[a_i, b_i[ \) is reduced to a point \( \tilde{a} \in \Delta \).

**Proposition 2.** A function \( g \) is the restriction of an FMF, that is \( g \) satisfies conditions (13) and (14), if and only if its form is given by
\[
g(x) = \begin{cases} 
\theta(x) & \text{if } x \in \Delta, \\
\tilde{a}_j & \text{if } x \in ]a_i, b_i[ \text{ and } \tilde{a}_j = \tilde{b}_j, \\
\tilde{F}_i^{-1}[c_i f_i(x)] & \text{if } x \in ]a_i, b_i[ \text{ and } \tilde{a}_j < \tilde{b}_j,
\end{cases}
\]  
(15)where \( \tilde{a}_j = g(a_i) \), \( \tilde{b}_j = g(b_i) \), \( c_i \) is a positive constant and \( \theta: \Delta \to \tilde{\Delta} \) is a non-decreasing function (with \( \theta(0) = 0 \), \( \theta(1) = 1 \)) subject only to conditions which ensure the global continuity of function \( g \).

**Proof.** For the sake of simplicity, the proof will be given in the case where \( I < +\infty \), so that \( \tilde{a}_j = \tilde{a}_i, \tilde{b}_j = \tilde{b}_i \). Nevertheless, it also holds in the general case (see [6]).

\( \Rightarrow \): It is evident that conditions (14) hold. With respect to (13), we can observe that, if \( x, y \) do not belong to the same open interval \( ]a, b[ \subseteq \Delta \), then (13) reduces to the identity \( \max[g(x), g(y)] = g[\max(x, y)] \), due to the monotonicity of \( g \). If both \( x, y \) belong to \( ]a_i, b_i[ \), then (13) is verified by the construction.

\( \Leftarrow \): Let \( \theta(x) = g|_\Delta(x) \) be the restriction of \( g \) to the idempotent set. Conditions on function \( \theta \) are obviously satisfied. Moreover if \( x \in \Delta \) (or \( y \in \Delta \)), then compatibility condition (13) is always satisfied since \( \theta(x) \in \tilde{\Delta} \) (or \( \theta(y) \in \tilde{\Delta} \)). If \( x \in ]a_i, b_i[ \), then \( g(x) \in ]\tilde{a}_i, \tilde{b}_i[ \) and therefore, by the continuity of functions \( g, f_i, \tilde{F}_i \), in a suitable open subset of \( ]a_i, b_i[ \times ]a_i, b_i[ \) Eq. (13) takes the form
\[
\tilde{F}_i^{-1}[\tilde{F}_i(g(x)) + \tilde{F}_i(g(y))] = g[F_i^{-1}(f_i(x) + f_i(y))].
\]  
(16)
By posing \( u = f_i(x) \), \( v = f_i(y) \), \( H(\xi) = f_i \circ g \circ f_i^{-1}(\xi) \), Eq. (16) may be rewritten as
\[
H(u + v) = H(u) + H(v).
\]
Its general solution is \( H(u) = f_i \circ g \circ f_i^{-1}(u) = c_i \cdot u \), from which we obtain \( g(x) = f_i^{-1}(c_i \cdot f_i(x)) \). Finally, if \( x, y \) belong to two disjoint open intervals of \( A^c \) and therefore \( g[F(x, y)] = \max\{g(x), g(y)\} = \max[g(x), g(y)] = \max[F(x), g(y)] \), No other conditions are imposed by the compatibility equation; so the theorem is completely proved. \( \square \)

3. One-valued fuzzy sets

In this paragraph we will determine the measure of a fuzzy set whose membership function assumes only one meaningful (non-null) value. Let \( \alpha_r \) be a membership value in \( L \) and let \( \tilde{\alpha} \) be a fuzzy subset of type
\[
\tilde{\alpha}(\omega) = \begin{cases} 
\alpha_r & \text{if } \omega \in A, \\
0 & \text{otherwise}.
\end{cases}
\]

The fuzzy measure of \( \tilde{\alpha} \) is given by
\[
m(\tilde{\alpha}) = \underbrace{G[m(A), \ldots, m(A), 0, \ldots, 0]}_{r \text{ times}} \overset{\text{def}}{=} g_r[m(A)].
\]

It is easy to prove, using the same arguments as in Section 2, that \( g_r(A) = \tilde{\alpha} \cap [0, g_r(1)] \), and that the function \( g_r \) has the following representation:

**Theorem 2.** The function \( g_r \) represents the measure of the one-level fuzzy set if and only if it has the following form:
\[
g_r(x) = \begin{cases} 
0_r(x) & \text{if } x \in A, \\
\tilde{\alpha}_{r,j} & \text{if } x \in ]a_i, b_i[ \text{ and } \tilde{\alpha}_{r,j} = \tilde{b}_{r,j}, \\
f_i^{-1}(c_i \cdot f_i(x)) & \text{if } x \in ]a_i, b_i[ \text{ and } \tilde{\alpha}_{r,j} < \tilde{b}_{r,j},
\end{cases}
\]

where \( \tilde{\alpha}_{r,j} = g_r(a_i), \tilde{b}_{r,j} = g_r(a_i), c_{r,i} \) is a positive constant and \( \theta_r : A \rightarrow \tilde{\alpha} \) is a non-decreasing function (with \( \theta_r(0) = 0, \theta_r(1) = 1 \)) subject only to conditions which ensure the global continuity of the function \( g_r \). Note that the function \( g \) in Section 2 coincides with the function \( g_n \). Moreover, as \( \tilde{m}(\alpha_r A) \leq \tilde{m}(\alpha_s A) \) when \( \alpha_r \leq \alpha_s \), the functions \( 0_r \), and the constants \( c_{r,i} \) have to be chosen so as to guarantee the condition
\[
\alpha_r \leq \alpha_s \Rightarrow g_r(x) \leq g_s(x).
\]

4. Two-valued fuzzy sets

In this section we will determine the measure of a fuzzy set whose membership function assumes only two meaningful (non-null) values \( \alpha_r \) (for some \( r < n \)) and \( \alpha_n = 1 \). This is fundamental because it will be the basis on which we realize a recursive process which will furnish, in the end, the measure of any finitely valued fuzzy subset. The fuzzy sets we deal with in this section are
\[
\tilde{\alpha}(\omega) = \begin{cases} 
\alpha_r & \text{if } \omega \in B, \\
1 & \text{if } \omega \in C, \\
0 & \text{otherwise},
\end{cases}
\]
where $B, C$ are crisp subsets of $\Omega$. In this case the two meaningful $x$-cuts of $\tilde{A}$ are $A_r = B \cup C$ and $A_n = C$. Now, let us introduce the function

$$G^*_r(x, y) = G(x, \ldots, x, y, \ldots, y).$$

It is easy to recognize that, by letting $x = m(A_r)$, $y = m(A_n)$, we have $\tilde{m}(\tilde{A}) = G^*_r(x, y)$ and moreover, by definition $G^*_r(y, y) = G(y, \ldots, y, y, \ldots, y) = g(y)$. The following conditions, as a consequence of (5)–(8), hold:

$$\text{Dom}(G^*_r) = \Gamma_2 = \{(x, y)|x \geq y \geq 0\},$$

$G^*_r$ is monotonic in both variables,

$$G^*_r(0, 0) = 0,$$

$$G^*_r(1, 1) = 1,$$

$$\tilde{F}[G^*_r(x, y), G^*_r(x', y')] = G^*_r[F(x, x'), F(y, y')].$$

From now on, the last relation will be indicated as “compatibility equation”.

Let $g(x), g_r(x)$ be two functions of type (15), (19), respectively, subject to the further conditions $g_r(x) \leq g(x)$, that is, $\theta_i(x) \leq \theta(x)$, $c_{r, i} \leq c_i$, $\tilde{a}_{r, j} \leq \tilde{a}_j$. Then the following representation holds:

**Theorem 3.** With the same notations as in Theorems 1 and 2, the general solution of system (21) is

$$G^*_r(x, y) = \begin{cases} 
\text{(a)} & \max[g_r(x), g(y)] & \text{if } x \in A \text{ or } y \in A, \\
\text{(b)} & \max[g_r(x), g(y)] & \text{if } x \in ]a_k, b_k[, \ y \in ]a_i, b_i[, \ k > i, \ g_r(a_k) \neq \tilde{a}_k, \\
\text{(c)} & f_j^{(k)}(c_i, f_i(x) + c_i f_i(y)) & \text{if } x \in ]a_k, b_k[, \ y \in ]a_i, b_i[, \ k > i, \ g_r(a_k) = \tilde{a}_k, \\
\text{(d)} & g(y) & \text{if } (x, y) \in ]a_i, b_i[^2, \ g_r(a_i) \neq \tilde{a}_i, \\
\text{(e)} & f_j^{(k)}(c_i, f_i(x) + (c_i - c_{r, i}) f_i(y)) & \text{if } (x, y) \in ]a_i, b_i[^2, \ g_r(a_i) = \tilde{a}_i. 
\end{cases}$$

We will prove only the necessity of this condition: “if $G^*_r$ satisfies the functional system (21), then it has the representation (22)”. The proof of the sufficiency consists only in a tedious, but easy verification of a wide range of possible cases corresponding to all possible choices of the values $x, y, x', y'$ in the last of the Eq. (21). Moreover we will consider, for the sake of simplicity, the case where $I < +\infty$, where $\tilde{a}_j = \tilde{a}_i$ and $\hat{f}_j = \hat{f}_i$. The proof in the general case may be found in [6].

**Case (a):** Suppose that $y \in A$. Then, by Proposition 1, $g(y) \in A$ (if $x \in A$ the proof is symmetric). Thus, the compatibility equation may be written as

$$G^*_r(x, y) = G^*_r[\max(x, y), \max(0, y)] = \max[G^*_r(x, 0), G^*_r(y, y)] = \max[g_r(x), g(y)].$$

**Cases (b, c):** In these cases $F(x, y) = \max(x, y)$. Thus, remembering that $x \geq y$ and by using the compatibility equation, we can write

$$G^*_r(x, y) = G^*_r[\max(x, y), \max(0, y)] = \tilde{F}[g_r(x), g(y)].$$

The proof can be completed by observing that the form (2) of the function $\tilde{F}$ is “max” when the arguments do not belong to the same interval $]a, b[$ as happens in case (b). On the other hand, $\tilde{F}$ is $f^{(1)}(\ldots)$ when, as in case (c), the arguments belong to the same interval. Moreover, we recall that the form of the functions $g_r, g$ is given by Propositions 1 and 2.
5. The measure of a fuzzy set

Now, by using the result of the previous section, we can install an iterative procedure which will give us
the general form of the fuzzy measure function \( G(x_1, \ldots, x_n) \). Intuitively speaking, the idea is the following.

Starting from a fuzzy subset \( \tilde{A} \) with membership values in \( L = \{ x_0 = 0, x_1, \ldots, x_n = 1 \mid x_i < x_{i+1} \} \), we construct
a sequence \( \{ \tilde{A}^{(t)} \mid i = 1, \ldots, n \} \) of fuzzy sets where \( \tilde{A}^{(1)} = \tilde{A} \), and \( \tilde{A}^{(t)} \) is obtained from \( \tilde{A}^{(t-1)} \) by suppression of
two \( z \)-cuts and insertion of a new \( z \)-cut, which is equivalent to the suppressed ones from the point of view
of the measure. More precisely, what we try to impose is that the fuzzy measures of each \( \tilde{A}^{(t)} \) are equal to the
fuzzy measure of \( \tilde{A} \). We stop the process when \( t = 1 \), that is when we have obtained a one-level fuzzy set.

\[ A + B = c_i. \]
Then we apply the result of Section 2 to obtain the measure of $\tilde{A}$. This could be interpreted as a procedure that (see Fig. 1).

- substitutes $\alpha$-cut $A_\alpha$ by a suitable subset $A^*$,
- destroys $A_{\alpha-1}$, thus eliminating the value $z_{n-1}$ in $L$,
- completes the $\alpha$-cut $A_{\alpha-2}$ by setting $\tilde{A}^{(n-1)}(\omega) = z_{n-2}$, $\forall \omega \in A_{\alpha-1} - A^*$.

Thus, we constructed a new fuzzy subset $\tilde{A}^{(n-1)}$ which has membership values in $L_{n-1} = L - \{z_{n-1}\}$. The idea is to organize the construction of $A^*$ in such a way that $\tilde{m}(\tilde{A}^{(n-1)}) = \tilde{m}(A)$. Then the procedure restarts, now with $n = n - 1$, and proceeds with diminishing $n$ at each step, until $n = 1$. Formally, we proceed as follows. We start by supposing that

$$G(x_1, \ldots, x_n) = G[x_1, \ldots, x_{n-2}, \gamma_n(x_{n-1}, x_n)],$$

(26)

that is we suppose that the function $G$ depends on its last two arguments only by means of a suitable real function of $x_{n-1}, x_n$. Although it seems natural in the procedure which constructs the measure level by level, we are really making a new restriction on the class of allowable measures. It looks like a kind of branching property. It is easy to check that the domain of $\gamma_n$ is the set

$$\Gamma_2 = \{(x, y) | x \geq y \geq 0\},$$

(27)

$$y \leq \gamma_n(x, y) \leq x.$$  

(28)

By using expression (26) we recognize that the first and second members of the compatibility equation (5) have the following form:

$$F[G(x_1, \ldots, x_n), G(y_1, \ldots, y_n)]$$

$$= F[G(x_1, \ldots, x_{n-2}, \gamma_n(x_{n-1}, x_n)), G(y_1, \ldots, y_{n-2}, \gamma_n(y_{n-1}, y_n))]$$

$$= G[F(x_1, y_1), \ldots, F(x_{n-2}, y_{n-2})] \gamma_n \{F(x_{n-1}, y_{n-1}), F(x_n, y_n)\}], \quad (29)$$

$$G[F(x_1, y_1), \ldots, F(x_n, y_n)]$$

$$= G[F(x_1, y_1), \ldots, F(x_{n-2}, y_{n-2}), \gamma_n(F(x_{n-1}, y_{n-1}), F(x_n, y_n))]. \quad (30)$$

By comparing these two expressions we can conclude that the compatibility equation is satisfied if (and only if in the case of strict monotonicity)

$$F[\gamma_n(x_{n-1}, x_n), \gamma_n(y_{n-1}, y_n)] = \gamma_n[F(x_{n-1}, y_{n-1}), F(x_n, y_n)]. \quad (31)$$

From this equation we can easily deduce that

$$\gamma_n(A) \subseteq A. \quad (32)$$

It suffices in (31) to set $x_{n-1} = y_{n-1} = a, x_n = y_n = b$ and suppose that $a, b \in A$. On the other hand, the function $G(x_1, \ldots, x_n) = G[x_1, \ldots, x_{n-2}, \gamma_n(x_{n-1}, x_n)]$ is an FMF. This implies that

- $\gamma_n$ is monotonic non-decreasing,
- $\gamma_n$ is continuous. \quad (33)

Eqs. (31)–(33) show that the function $\gamma_n$ satisfies all the conditions on the function $G^*_\alpha$ given in Section 4, and therefore it can be determined by the procedure in the same section. Its general form is given by formula
(22) with two slight differences: (i) \( \tilde{f}_j, \tilde{a}_j \) have to be substituted by \( f_j, a_j \), since in the actual compatibility equation (31) \( \tilde{F} \) has been changed to \( F \), (ii) \( g(x) \) becomes the identity, since now, from (28), \( \gamma_n(x, x) = x \). So the form of \( \gamma_n \) is

\[
\gamma_n(x, y) = \begin{cases} 
(a) \max[h_n(x), y] & \text{if } x \in A \text{ or } y \in A, \\
(b) \max[h_n(x), y] & \text{if } x \in [a_k, b_k[, y \in ]a_i, b_i[, k > i, \quad h_n(a_k) \neq a_j, \\
(c) f_j^{-1}(c_{n,k} f_k(x) + c_i f_i(y)) & \text{if } x \in ]a_k, b_k[, y \in ]a_i, b_i[, k > i, \quad h_n(a_k) = a_j, \\
(d) y & \text{if } (x, y) \in ]a_i, b_i[^2, \quad h_n(a_i) \neq a_j, \\
(e) f_j^{-1}(c_{n,i} f_i(x) + (c_i - c_{n,i}) f_i(y)) & \text{if } (x, y) \in ]a_i, b_i[^2, \quad h_n(a_i) = a_j, 
\end{cases}
\]

(34)

where \( h_n(x) = \gamma_n(x, 0) \). Note that what we did here has been to transform the computation of the measure of a fuzzy set with \( n \) allowable membership values \( \{z_1, \ldots, z_{n-1}, 1\} \) into the computation of the measure of an equivalent fuzzy subset with \( n - 1 \) membership values \( \{z_1, \ldots, z_{n-2}, 1\} \), by using the function \( \gamma_n \). The iteration \((n - 1)\) times of this operation leads us to an equivalent set with only one membership value \( z = 1 \), that is a crisp subset. The fuzzy measure of this subset can be obtained by means of the function \( g \) in Section 2. So, by setting \( x_i = m(A_i) \), we finally obtain

\[
\tilde{m}(\tilde{A}) = g(\gamma_2(x_1, \gamma_3(x_2, \gamma_4(x_3, \ldots \gamma_n(x_{n-1}, x_n) \ldots))).
\]

(35)

6. Two examples

In the examples we present here the set of idempotents is \( A = [0, a] \cup [b, 1] \), and contains two particular important subcases:

- \( b = 1 \), that is \( A = [0, a] \cup \{1\} \). In this case the measures less than \( a \) are negligible in the sense that as soon as there exists a measure \( m(A_i) \) greater than \( a \), the \( x \)-cuts \( A_i \) with \( m(A_i) \leq a \) are discarded.
- \( a = 0 \), that is \( A = \{0\} \cup [b, 1] \). In this case the measures greater than \( b \) are dominant in the sense that as soon as there exists a measure \( m(A_i) \) greater than \( b \), the \( x \)-cuts \( A_i \) with \( m(A_i) \leq b \) are discarded and the global measure becomes \( \max \{m(A_i)\} \).

Example 1. We can choose the same function \( G \) to construct all the two level’s set measures, that is \( G^* = G \), \( \forall i \) with, \( G(a, 0) = a \), \( G(b, 0) = b \) and \( G(1, 0) = (b + 1)/2 \). Let us select the additive generators of \( m, \tilde{m} \) in \([a, b[\) equal to \(- \log(\alpha \beta/(b - a))\) and \( x \), respectively. A choice of \( G(x, 0) \) compatible with the results of Proposition 5 is

\[
G(x, 0) = \begin{cases} 
    x & \text{if } x \leq a, \\
    b - \frac{(b - x)\alpha}{(b - a)\gamma - 1} & \text{if } a < x < b, \\
    \frac{x + b}{2} & \text{if } x \geq b.
\end{cases}
\]

(36)
The result of Propositions (8), (9) assigns to the function $G$ the following form:

$$
G(x, y) = \begin{cases} 
  x & \text{if } (x, y) \in [0, a] \times [0, a], \\
  \frac{b + 1}{2} & \text{if } x \in [b, 1] \times [0, a], \\
  b - (b - x)(b - y)^{1-c} & \text{if } x \in ]a, b[ \times ]a, b[, \\
  b - \frac{(b - x)}{(b - a)^{1-c}} & \text{if } x \in ]a, b[ \times [0, a], \\
  \frac{x + b}{2} & \text{if } x \in [b, 1] \times [b, 1]. 
\end{cases}
$$

(37)

Example 2. We choose $G(a, 0) = a$, $G(b, 0) = b$, $G(1, 0) = 1$ and

$$
G_2(x, 0) = \begin{cases} 
  x & \text{if } x \in A, \\
  f^{(-1)}[c_2 f(x)] & \text{if } a < x < b, 
\end{cases}
$$

(38)

where $f$ is any additive generator, and $c_2 > 0$. From Propositions (8), (9) and formula (27) we obtain

$$
G(x, y) = \begin{cases} 
  \theta_2(x) = x & \text{if } x \in A, \\
  f^{(-1)}[c_2 f(x)] & \text{if } a < x < b, y \in A, \\
  f^{(-1)}[c_2 f(x) + (1 - c_2) f(y)] & \text{if } (x, y) \in ]a, b[^2. 
\end{cases}
$$

(39)

In order to obtain the form of the measure of a multi-level fuzzy subset we assume that all the maps $G_i^*$ have the same form of $G$, possibly with different constants $c_i$, and $\theta_i(x) = x$. The general form of measure $\tilde{m}$ is obtained by means of the following function $G(x_1, \ldots, x_n)$:

$$
G(x_1, \ldots, x_n) = \begin{cases} 
  g(x_1) & \text{if } x_1 \in A, \\
  g \circ f^{(-1)} \left[ \sum_{i=1}^{k} \prod_{j=2}^{k} (1 - c_j) c_{i+1} f(x_i) \right] & \text{otherwise.} 
\end{cases}
$$

(40)

where $k = \max\{j | x_j \in ]a, b[\}$. Moreover, $c_{n+1} = 1$ by definition.

References


